A p-adic quantum group and the quantized p-adic upper half plane

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Christian Wald

Präsidentin der Humboldt-Universität zu Berlin:
Prof. Dr.-Ing. Dr. Sabine Kunst

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät:
Prof. Dr. Elmar Kulke

Gutachter:
1. Prof. Dr. Elmar Große-Klönne
2. Prof. Dr. Joachim Mahnkopf
3. Prof. Dr. Tobias Schmidt

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Summary

A quantum group is a noncommutative noncocommutative Hopf algebra. Let $K \supseteq \mathbb{Q}_p$ be a finite extension with valuation ring $\mathcal{O}$. Then the algebra of locally analytic functions on $\mathrm{GL}(2, \mathcal{O})$ is a locally convex K-Hopf algebra. In this thesis we deform this Hopf algebra and obtain a locally convex noncommutative noncocommutative K-Hopf algebra $C_q^{la}(\mathrm{GL}(2, \mathcal{O}), K)$, i.e. a p-adic quantum group. We show that the algebra of quantum locally analytic distributions $D_q(\mathrm{GL}(2, \mathcal{O}), K) := C_q^{la}(\mathrm{GL}(2, \mathcal{O}), K)'_b$ is a Fréchet Stein algebra.

An important class of locally analytic representations of $\mathrm{GL}(2, K)$ is constructed from global sections on the p-adic upper half plane. We construct a quantized analogue of the p-adic upper half plane which we hope will give rise to a locally analytic admissible representation of our p-adic quantum group.

In chapter 2 we construct $C_q^{la}(\mathrm{GL}(2, \mathcal{O}), K)$ using the quantum matrix algebra $M_q(2, K)$ and multiplicative valuations on it. We show that $C_q^{la}(\mathrm{GL}(2, \mathcal{O}), K)$ is a noncommutative locally convex K-Hopf algebra that is of compact type.

In chapter 3 we investigate $D_q(\mathrm{GL}(2, \mathcal{O}), K)$. Therefore we describe a subalgebra of $D_q(\mathrm{GL}(2, \mathcal{O}), K)$ as a completion of the quantum enveloping algebra $U_q(\mathfrak{gl}_2, K)$. This subalgebra, which we denote by $D_q^m(e, r)$, is constructed by using partial divided powers and we show that it is Noetherian. We show that for certain $m_1, m_2 \in \mathbb{N}$ and $r_1, r_2 \in \mathbb{Q}$ there are canonical K-Banach algebra morphisms

$$D_q^{m_2}(e, r_2) \longrightarrow D_q^{m_1}(e, r_1)$$

which are right flat. This enables us to show that $D_q(\mathrm{GL}(2, \mathcal{O}), K)$ is a Fréchet Stein algebra.

We construct an analogue of the p-adic upper half plane in chapter 4. The key ingredients for this construction are the Bruhat-Tits tree of $\mathrm{PGL}(2, K)$, the Manin quantum plane $K[x, y]_q := K\{x, y\}/(xy - qyx)$ and the theory of algebraic microlocalization.

Zusammenfassung

Eine Quantengruppe ist eine nicht kommutative und nicht kokommutative Hopfalgebra. Sei $K \supseteq \mathbb{Q}_q$ eine endliche Körpererweiterung mit Bewertungsring $\mathcal{O}$. Die Algebra der lokalanalytischen Funktionen auf $\mathrm{GL}(2, \mathcal{O})$ ist eine lokalkonvexe K-Hopfalgebra. In der vorliegenden Arbeit deformieren wir diese Hopfalgebra und erhalten eine lokalkonvexe K-Hopfalgebra $C_q^{la}(\mathrm{GL}(2, \mathcal{O}), K)$, die weder kommutativ noch kokommutativ ist. Dies ist ein Beispiel einer p-adischen Quantengruppe. Wir zeigen dann, dass die Distributionenalgebra $D_q(\mathrm{GL}(2, \mathcal{O}), K) := C_q^{la}(\mathrm{GL}(2, \mathcal{O}), K)'_b$ eine Fréchet Stein Algebra ist.
Eine wichtige Klasse lokalanalytischer Darstellungen von $\mathrm{GL}(2, K)$ tritt als Linienbündel auf der p-adischen oberen Halbebene auf. Wir konstruieren eine quantisiertes Pendant zur p-adischen oberen Halbebene und hoffen, dass dieses eine zulässige Darstellung unserer p-adischen Quantengruppe induziert.
In Kapitel 2 benutzen wir die Quantenmatrixalgebra und Bewertungen auf derselben um $C_q^{la}(\mathrm{GL}(2, \mathcal{O}), K)$ zu konstruieren.
Wir zeigen, dass $C_q^{la}(\mathrm{GL}(2, \mathcal{O}), K)$ eine nicht kommutative lokalkonvexe K-Hopfalgebra von kompaktem Typ ist.
In Kapitel 3 untersuchen wir die Distributionenalgebra $D_q(\mathrm{GL}(2, \mathcal{O}), K)$. Dazu beschreiben wir eine Unteralgebra von $D_q(\mathrm{GL}(2, \mathcal{O}), K)$ als Vervollständigung der quantisierten universellen Einhüllenden $U_q(\mathfrak{gl}_2, K)$. Diese Unteralgebra, für welche wir $D_q^m(e, r)$ schreiben, wird unter der Benutzung von partiell dividierten Potenzen konstruiert und wir können zeigen, dass sie noethersch ist. Wir zeigen, dass für gewisse $m_1, m_2 \in \mathbb{N}$ und $r_1, r_2 \in \mathbb{Q}$ kanonische K-Banach Algebramorphismen

$$D_q^{m_2}(e, r_2) \longrightarrow D_q^{m_1}(e, r_1)$$

existieren, welche rechtsflach sind. Dies versetzt uns in die Lage zu zeigen, dass $D_q(\mathrm{GL}(2, \mathcal{O}), K)$ eine Fréchet Stein Algebra ist.
Das Pendant zur p-adischen oberen Halbebene konstruieren wir in Kapitel 4. Hierfür benutzen wir den Bruhat-Tits Baum von $\mathrm{PGL}(2, K)$, die Manin Quantenebene $K[x, y]_q := K\{x, y\}/(xy - qyx)$ und die Theorie der algebraischen Mikrolokalisierung.

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Introduction

Locally analytic representations of p-adic groups play an important role in the p-adic local Langlands programm and in the study of p-adic symmetric spaces. These representations were studied by P. Schneider and J. Teitelbaum in a systematic way. A key notion in their studies is the algebra of locally analytic functions $C^{la}(G, K)$ on a p-adic group G over K. Here K is a finite extension of $\mathbb{Q}_p$. In[ST02] they made the observation that in order to study locally analytic representations one can also study continuous modules over the algebra of locally analytic distributions $D(G, K)$, which is the strong dual of $C^{la}(G, K)$.

In [ST03] P. Schneider and J. Teitelbaum introduced the notion of a Fréchet Stein algebra in order to overcome the problem that the category of locally analytic representations is in general extremely huge. In the same article they showed that for a compact p-adic group G, the algebra $D(G, K)$ is a Fréchet Stein algebra.

To obtain a category that is better behaved than the category of locally analytic representations they defined in [ST03] the category of coadmissible $D(G, K)$-modules.

A locally convex K-algebra A is called Fréchet Stein algebra if there exist Noetherian K-Banach algebras A_n with right flat transition maps $A_{n+1} \to A_n$ such that $A = \varprojlim_n A_n$. An A-module M is called coadmissible if there exists a system M_n of finitely generated A_n modules together with isomorphisms $A_n \otimes_{A_{n+1}} M_{n+1} \cong M_n$ and

$$M = \varprojlim_n M_n.$$

Thus the category of coadmissible $D(G, K)$-modules is of purely algebraic nature. It also has many other desirable features see e.g. [ST03]. It is now known that many important locally analytic representations are coadmissible $D(G, K)$-modules, see e.g. [STP01] and [PSS15].

In [Soi08] Y. Soibelman defines a notion of a compact p-adic quantum group and conjectures that the algebra of quantum locally analytic distributions is a Fréchet Stein algebra. Our approach to p-adic quantum groups differs from the one of Soibelman, but at the end the definitions turn out to be very simi-

lar. A p-adic quantum group is a noncommutative noncocommutative locally convex K-Hopf algebra. Let $\mathcal{O}$ be the valuation ring of K. From now on we write H for $\mathrm{GL}(2, \mathcal{O})$.
Since for a compact p-adic group G the algebra $C^{la}(G, K)$ is a locally convex K-Hopf algebra, a natural candidate for a p-adic quantum group is a deformation of $C^{la}(H, K)$. We construct such a deformation and show in chapter 3 that the algebra of quantum locally analytic distributions for this deformation is a Fréchet Stein algebra.

The aim of chapter 2 is the construction of a deformation of $C^{la}(H, K)$. We will start with the quantum matrix algebra $M_q(2, K)$ which is a noncommutative deformation of the matrix algebra depending on an element $q \in K^\times$. Since $M_q(2, K)$ has very few K-rational points, we cannot attach a meaningful p-adic manifold to $M_q(2, K)$. But we are still able to define a notion of power series on a disc.
Let ν be the valuation on K. For $g \in H$ we denote the disc around g with radius r by $B(g, r) \subseteq \mathcal{O}^4$. We will show that if $2r < \nu(1-q)$ we can attach to $B(g, r)$ a multiplicative valuation $\nu_{g,r}$ on $M_q(2, K)$, which enables us to define the algebra of converging (noncommutative) power series $C_q^{an}(g, r)$ on $B(g, r)$ as the completion of $M_q(2, K)$ with respect to $\nu_{g,r}$. Since $C_q^{an}(g, r)$ is noncommutative these power series do not define functions with values in K. We then show that for a covering $H = \coprod_i B(g_i, r)$ the space

$$C_q^{la}(H, r) := \bigoplus_i C_q^{an}(g_i, r)$$

is a K-Banach Hopf algebra. We define the p-adic quantum group

$$C_q^{la}(H, K) := \varinjlim_{2r < \nu(1-q)} C_q^{la}(H, r)$$

where r in the limit is increasing since we work with valuations instead of norms.
We show that the K-algebra $C_q^{la}(H, K)$ is a noncommutative and noncocommutative locally convex K-Hopf algebra and $C_1^{la}(H, K) = C^{la}(H, K)$, and thus we have constructed the desired p-adic quantum group.

In chapter 3 we will show that the strong dual of $C_q^{la}(H, K)$ which we denote by $D_q(H, K)$ is a Fréchet Stein algebra. In [ST03] P. Schneider and J. Teitelbaum prove that $D(G, K)$ is a Fréchet Stein algebra for a compact p-adic group G. Their proof uses the theory of Mahler series and K-points of the group G. Neither is the theory of Mahler series developed in the quantized case nor does $M_q(2, K)$ have enough K-points, and therefore their strategy does not seem to be promising in our case.

Fortunately in [Eme11] M. Emerton gives another proof of the Fréchet Stein property of $D(G,K)$ which does not depend on Mahler series. He uses a decreasing sequence of open compact subgroups G_r and analyzes the strong dual $D^{an}(G_r,K)$ of the analytic functions on such a subgroup. He then shows that for certain elements $\delta_{g_{i,r}} \in D(G,K)$ one has

$$D(G,K) = \varprojlim_{r<\infty} \left(\bigoplus_i \delta_{g_{i,r}} D^{an}(G_r,K) \right).$$

Note that since we use valuations rather than norms, the r in the limit is increasing. The algebras $D^{an}(G_r,K)$ can be described as completions of the universal enveloping algebra of the Lie algebra of G.
He then uses the technique of partial divided powers to define K-Banach algebras $D^m(G_r,K) \subseteq D^{an}(G_r,K)$. With the help of the algebras $D^m(G_r,K)$, he can show that $D(G,K)$ is a Fréchet Stein algebra by using filtration techniques for $D^m(G_r,K)$.

Since there exists a notion of the quantum enveloping algebra for $\mathrm{GL}(2,K)$, the approach of M. Emerton is much more well suited for our setting. For the proof that $D_q(H,K)$ is a Fréchet Stein algebra we will mostly follow his strategy.
Let $e \in H$ be the identity element. Recall that to a disc $B(e,r)$, which is a subgroup of $\mathrm{GL}(2,\mathcal{O})$, we attached the algebra $C_q^{an}(e,r)$, which is in fact a K-Banach Hopf algebra. Its strong dual $D_q^{an}(e,r) := C^{an}(e,r)'_b$ is a K-Banach subalgebra of $D_q(H,K)$. In order to see that $D_q^{an}(e,r)$ is a completion of the quantum enveloping algebra $U_q(\mathfrak{gl}_2,K)$ we will have to analyze a bracket

$$U_q(\mathfrak{gl}_2,K) \times M_q(2,K) \to K$$

which will be done in section 3.2.2.
We then define a K-Banach subspace $D_q^m(e,r) \subseteq D_q^{an}(e,r)$ using partial divided powers. In contrast to the case of p-adic groups it is not obvious that $D_q^m(e,r)$ is in fact a K-Banach algebra. Using filtration techniques we will show that for suitable $r_1, r_2 \in \mathbb{Q}$ with $r_2 \geq r_1$ and $m_1, m_2 \in \mathbb{N}$ the map

$$D_q^{m_2}(e,r_2) \longrightarrow D_q^{m_1}(e,r_1) \tag{0.0.1}$$

is a right flat map of Noetherian K-Banach algebras. In contrast to the case of p-adic groups, we can't restrict ourselves to the case where $r_1, r_2 \in \nu(K)$.
In section 3.4 we will show that we can find elements $\delta_{g_{i,r_n}} \in D_q(H,K)$ and a sequence $(r_n, m_n)_{n\in\mathbb{N}} \subset (\mathbb{Q}\times\mathbb{N})^{\mathbb{N}}$ such that $\bigoplus_i \delta_{g_{i,r_n}} D_q^{m_n}(e,r_n)$ is a Noetherian

K-Banach subalgebra of the locally convex K-algebra $D_q(H,K)$ and that

$$D_q(H,K) = \varprojlim_n \left(\underset{i}{\oplus} \delta_{g_{i,r_n}} D_q^{m_n}(e, r_n) \right).$$

Using (0.0.1) we will conclude that $D_q(H,K)$ is a Fréchet Stein algebra.

An important class of coadmissible $D(H,K)$-modules is constructed from global sections of line bundles on the p-adic upper half plane $\mathcal{H}$, see e.g. [DT08] and [STP01]. Thus it would seem natural to try to construct coadmissible $D_q(H,K)$-modules by constructing a quantization of the p-adic upper half-plane.

In chapter 4 we construct a quantized analogue of the p-adic upper half plane and we will now briefly describe what kind of object this analogue is. Recall that there is a reduction map

$$r : \mathcal{H} \longrightarrow \mathcal{T}$$

to the Bruhat-Tits tree of $\mathrm{PGL}(2,K)$. For a finite subtree $\mathcal{S} \subseteq \mathcal{T}$ we have that $r^{-1}(\mathcal{S})$ is affinoid. Thus we can associate to $\mathcal{S}$ the K-Banach algebra $\mathcal{O}_{\mathcal{H}}\left(r^{-1}(\mathcal{S})\right)$.

We will define an infinite subtree $\mathcal{T}_q \subseteq \mathcal{T}$. To every subtree $\mathcal{S} \subseteq \mathcal{T}_q$ we will attach a locally convex K-algebra $\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$, which is a K-Banach algebra if $\mathcal{S}$ is finite. Let $\mathrm{Spmqb}\left(\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})\right)$ be the space of continuous quasi abelian semivaluations on $\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$. For every finite subtree $\mathcal{S} \subseteq \mathcal{T}_q$ we will define a reduction map

$$r : \mathrm{Spmqb}\left(\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})\right) \longrightarrow \mathcal{S}.$$

The assignment $\mathcal{S} \mapsto \mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$ is our quantized p-adic upper half plane.

Now we will briefly describe how this assignment is constructed. Since the p-adic upper half plane $\mathcal{H} := \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(K)$ is one dimensional, the algebras $\mathcal{O}_{\mathcal{H}}\left(r^{-1}(\mathcal{S})\right)$ are completions of localizations of $K[x]$. But $K[x]$ doesn't seem to have an obvious quantization.

Therefore we replace $\mathcal{H}$ with the two dimensional analogue

$$\mathcal{A} := \mathbb{A}^2(\mathbb{C}_p) - \{K\text{-rational hyperplanes containing } 0\}.$$

Here the affinoids are completions of localizations of $K[x,y]$. With the Manin quantum plane $K[x,y]_q := K\{x,y\}/(xy - qyx)$ there already is a quantization of $K[x,y]$ and we will use it as a starting point. As before we have a reduction map

$$r : \mathcal{A} \longrightarrow \mathcal{T}.$$

Let $\mathcal{S}$ be a finite subtree. Then $\mathcal{O}_{\mathcal{A}}\left(r^{-1}(\mathcal{S})\right)$ is a completion of a localization of $K[x,y]$. To every subtree $\mathcal{S} \subseteq \mathcal{T}_q$ we will attach a noncommutative complete K-algebra $\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S})$. We construct $\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S})$ out of $K[x,y]_q$ using the theory of algebraic microlocalization developed by P. Schneider in [Záb12]. We will then single out a complete K-subalgebra $\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$ of $\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S})$ which is an analogue of $\mathcal{O}_{\mathcal{H}}\left((r^{-1}(\mathcal{S})\right)$. For a finite subtree $\mathcal{S} \subset \mathcal{T}_q$ the algebra $\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$ is a K-Banach algebra.
Although not part of this thesis it can be shown that there exists a finite subtree $\mathcal{T}_a \subset \mathcal{T}_q$ such that $\mathcal{O}_{\mathcal{T}_q}(\mathcal{T}_a)$ is a topological $C_q^{la}(H,K)$ comodule. Our conjecture is, that this comodule gives rise to a coadmissble $D_q(H,K)$-module.

There is another approach to noncommutative analytic spaces over nonarchimedean fields by Y. Soibelman developed in [Soi09]. He does not work with quasi abelian multiplicative semivaluations but with an analogue of Rosenbergs spectrum of an abelian category. The spectrum he defines contains all the quasi abelian multiplicative semivaluations but it is in general much larger.
Up to now there does not seem to be a good way of attaching algebras to certain subsets of his spectrum. Our example $\mathcal{O}_{\mathcal{T}_q}$ suggests that using quasi abelian multiplicative semivaluations and algebraic microlocalization should play a role in the attempt to overcome this problem.

Notation

Unless otherwise stated K will be a finite extension of $\mathbb{Q}_p$ for a prime number $p > 2$. The valuation on K is denoted by ν and the valuation ring is denoted by $\mathcal{O}$. We fix a uniformizer $\pi \in \mathcal{O}$ and assume that ν is normalized i.e. $\nu(\pi) = 1$. We write e for the ramification index $\nu(p)$. The residue field $\mathcal{O}/(\pi)$ will be denoted by κ.

If we have a radius r we always assume that $r \in \mathbb{Q}$. For a natural number $n \in \mathbb{N}_0$ we will write $S(z)$ for the sum of its p-adic digits. We then have that $\nu(n!) = \frac{e(n-S(z))}{p-1}$. For an element $\mu = (\mu_1, ..., \mu_n) \in \mathbb{Z}^n$ we define $|\mu| = \sum_i |\mu_i|$. If we have a function $f : X \to X$ we will write f^n for the n-th iterate $f \circ \cdots \circ f$ of f. We will write $\lceil \cdot \rceil$ for the ceiling function and $\lfloor \cdot \rfloor$ for the floor function. For two elements $a, b \in A$ of a ring A we write $[a,b] = ab - ba$ for their commutator.

Chapter 1

Some non-archimedean functional analysis

In this chapter we recall some fundamental facts about nonarchimedean functional analysis that can be found for example in [Sch02]. Moreover we will recall some definitions concerning locally convex K-Hopf algebras. The main purpose of this chapter is to provide a reference for the upcoming chapters.

1.1 Short review of locally convex K-vector spaces

Let K be a discretely valued complete field, $\mathcal{O}$ its valuation ring with normalized valuation ν and residue field κ.

Definition 1.1.1. A lattice in a K-vector space V is an $\mathcal{O}$-submodule W such that $K \otimes_{\mathcal{O}} W \to V$ is surjective. Let V be a topological K-vector space. Assume that a basis of its topology is given by a family $\mathfrak{L}$ of lattices such that

1. For $L \in \mathfrak{L}$ and $a \in K^{\times}$ there exists an $L' \in \mathfrak{L}$ such that $L' \subset aL$
2. For $L_1, L_2 \in \mathfrak{L}$ there exists an $L_3 \in \mathfrak{L}$ such that $L_3 \subseteq L_1 \cap L_2$.

Then V is called locally convex K-vector space. A locally convex K-vector space V is called Fréchet space if its topology can by defined by a countable set of open lattices.

Definition 1.1.2. Throughout this text we will use the notion of valuations rather than the notion of norms. Let V be a K vector space. A semivaluation on V is a function $\nu_V : V \to \mathbb{R} \cup \{\infty\}$ with the property that

1. $\nu_V(0) = \infty$;
2. $\nu_V(v + w) \geq \min\{\nu_V(v) + \nu_V(w)\}$;
3. $\nu_V(av) = \nu(a) + \nu_V(v)$ for all $a \in K$, $v \in V$.

If in addition $\nu_V(v) = \infty$ implies that $v = 0$ then ν_V is called valuation. A K-Banach space is a K-vector space that is complete with respect to some valuation. Let A be a K-algebra with a valuation ν_A. Then ν_A is called submultiplicative if

$$\nu_A(ab) \geq \nu_A(a) + \nu_A(b)$$

for all $a, b \in A$. A K-Banach algebra is a K-algebra A with a submultiplicative valuation ν_A such that A is complete with respect to ν_A. For a K-algebra A with submultiplicative valuation ν_A its completion $\hat{A}$ is a K-Banach algebra. If (A, ν_A) is a K-Banach algebra and $I \subset A$ is a proper closed two sided ideal, then there is a canonical submultiplicative residue valuation on A/I given by

$$\nu_{A/I}(a + I) = \sup\{\nu_A(a + i) : i \in I\}.$$

Then $(A/I, \nu_{A/I})$ is a K-Banach algebra, see e.g. [Ber90] 1.1.1 (iii).

Definition 1.1.3. For a locally convex K-vector space V let V' be the subspace of the linear dual consisting of continuous functions. There are several possibilities to endow it with a topology. We will always consider the strong topology which is the topology of uniform convergence on bounded subsets, see e.g. [Sch02] chapter 6. We will denote the K-vector space V' together with the strong topology by V'_b. If V is a K-Banach space with valuation ν_V then the topology on V'_b is given by the valuation $\nu_{V'_b}$ defined by

$$\nu_{V'_b}(\lambda) := \inf \left\{\nu(\lambda(v)) - \nu_V(v) : v \in V \backslash \{0\}\right\}$$

for $\lambda \in V'_b$.

Definition 1.1.4. For an index set H let $(V_h)_{h \in H}$ be a family of locally convex K-vector spaces. Let V be a K-vector space and assume we have linear maps $f_h : V_h \longrightarrow V$. Then there is a unique finest locally convex topology on V such that all f_h are continuous. It is called the locally convex final topology with respect to $(f_h)_{h \in H}$. For an inductive system $(V_h)_{h \in H}$ of locally convex K-vector spaces we define the locally convex inductive limit $\varinjlim_h V_h$ to be the usual inductive limit of vector spaces together with the locally convex final topology.

Proposition 1.1.5. *[Sch02] 5.1. Assume that V carries the locally convex final topology with respect to a family of linear maps $f_h : V_h \to V$. Assume that the topology on V_h is defined by the family of lattices $(L_{h_j})_{j \in J(h)}$. Assume furthermore that $V = \sum_{h \in H} f_h(V_h)$. Then the topology on V is defined by the family of lattices*

$$\left\{ \sum_{h \in H} f_h \left(L_{h_{j(h)}} \right) : j(h) \in J(h) \right\}.$$

Definition 1.1.6. For an index set H let $(V_h)_{h\in H}$ be a family of locally convex K-vector spaces. Let V be a K vector space and assume we have maps $f_h : V \longrightarrow V_h$. The unique coarsest locally convex topology on V for which all f_h are continuous is called the initial topology with respect to $(f_h)_{h\in H}$. For a projective system $(V_h)_{h\in H}$ of locally convex K-vector spaces we define the locally convex projective limit $\varprojlim_h V_h$ to be the usual projective limit of vector spaces together with the initial topology.

Definition 1.1.7. Let V be a locally convex K-vector space. A subset $A \subseteq V$ is called bounded if for any open lattice $L \subseteq V$ there exists $a \in K$ such that $A \subseteq aL$.

Definition 1.1.8. Let V be a locally convex K-vector space. A bounded $\mathcal{O}$-submodule $A \subseteq V$ is called compactoid if, for any open lattice $L \subseteq V$ there exist finitely many vectors $v_1, ..., v_n$ such that $A \subseteq L + \mathcal{O}v_1 + \cdots + \mathcal{O}v_n$.

Definition 1.1.9. A continuous linear map $f : V \longrightarrow W$ between locally convex K-vector spaces is called compact if there exists an open lattice $L \subseteq V$ such that the closure of $f(L)$ in W is compactoid.
A locally convex vector space V is called of compact type if it is the locally convex inductive limit $V \cong \varinjlim_n V_n$ of K-Banach spaces with injective compact transition maps.

Lemma 1.1.10. *Let $r \in \mathbb{R}, l \in \mathbb{N}$ and let*

$$V_{r,l} := \left\{ \{a_n\}_{n\in\mathbb{N}^l} \in K^{\mathbb{N}^l} : \lim_{|n|\to\infty} (\nu(a_n) + r|n|) = \infty \right\}.$$

with valuation $\nu_{r,l}\left(\{a_n\}_{n\in\mathbb{N}^l}\right) := \min\{\nu(a_n) + r|n| : n \in \mathbb{N}^l\}$. Then

1. *$V_{r,l}$ is a K-Banach space for all $l \in \mathbb{N}$ and $r \in \mathbb{R}$.*

2. *For $r < r'$ and $l \in \mathbb{N}$ the inclusion $V_{r,l} \subseteq V_{r',l}$ is a continuous, injective and compact map.*

Proof. 1. is obvious. For 2. one can use a similar proof as in [Mor81] Lemma 3.5. We will give a proof adapted to our situation.
The continuity is immediate from the definition of the valuation and we will now show the compactness. Let $L := \{v \in V_{r,l} : \nu_{r,l}(v) \geq 0\}$. Let $\overline{L}$ be the closure of L in $V_{r',l}$. Then

$$\overline{L} = \{\{a_n\}_{n\in\mathbb{N}^l} : \nu(a_n) + r|n| \geq 0\}.$$

For $j \in \mathbb{N}$ let $M_j := \left\{v \in V_{r',l} : \nu_{r',l}(v) \geq j\right\}$ and let $k_j \in \mathbb{N}$ be such that $k_j(r' - r) \geq j$ and let $\{a_n\}_{n\in\mathbb{N}^l} \in \overline{L}$. Then

$$\nu(a_n) + |n|r' = \nu(a_n) + |n|r + |n|(r' - r) \geq |n|(r' - r).$$

But this means that for $\{b_n\}_{n\in\mathbb{N}^l}$ with

$$b_n = \begin{cases} 0 \text{ if } |n| \le k_j \\ a_n \text{ if } |n| > k_j \end{cases}$$

we have that $b_n \in M_j$. Denote by $\pi_i^{-\lfloor ri \rfloor} \in V_{r',l}$ the sequence with entry $\pi^{-\lfloor ri \rfloor}$ at $i \in \mathbb{N}$ and entry 0 for all other indices. Then $a_n \in \sum_{|i|\le k_j} \mathcal{O}\pi_i^{-\lfloor ri \rfloor} + M_j$ and thus

$$\overline{L} \subseteq \sum_{|i|\le k_j} \mathcal{O}\pi_i^{-\lfloor ri \rfloor} + M_j.$$

Since for every open lattice N of $V_{r',l}$ there exists a $j \in \mathbb{N}$ such that $M_j \subseteq N$ we have shown that $\overline{L}$ is compactoid and thus the inclusion $V_{r,l} \subseteq V_{r',l}$ is compact. □

Lemma 1.1.11. *Let V_i, W_i be K-Banach spaces for $i \in \{1, ..., n\}$. Let products of K-Banach spaces be endowed with the product valuation.*

1. *Let $f_i : V \to V_i$ be compact maps of K-Banach spaces. Then also*

$$\prod_i f_i : V \to \prod_i V_i$$

   is compact.

2. *Let $f_i : V_i \to W_i$ be compact maps of K Banach spaces. Then also*

$$\prod_i f_i : \prod_i V_i \to \prod_i W_i$$

   is compact.

Proof. Let $L_i \subseteq V$ be a lattice such that the closure of $f_i(L_i)$ is compactoid. Then it is easy to see that closure of the image of $\cap_i L_i$ under $\prod_i f_i$ is compactoid and thus the first claim is true. The second is proven similarly. □

Definition 1.1.12. Let V be a locally convex K-vectors space. V is called bornological if every lattice $L \subseteq V$ with the property that for every bounded subset $B \subseteq V$ there exists an $a \in K$ such that $B \subseteq aL$, is open.

Proposition 1.1.13. *[Sch02] 16.10 Let*

$$V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow V_4 \longrightarrow \cdots$$

be an inductive system of Hausdorff locally convex K-vector spaces with injective and compact transition maps. We then have:

1. $\varinjlim_n V_n$ *is reflexive, bornological and complete;*

2. $\left(\varinjlim_n V_n\right)'_b$ *is a Fréchet space;*

3. *the map* $\left(\varinjlim_n V_n\right)'_b \longrightarrow \varprojlim_n (V_n)'_b$ *is a topological isomorphism.*

1.1.14. For locally convex K-vector spaces there are in general two notions of completed tensor product depending if one completes with respect to the inductive or the projective tensor product topology. But for Fréchet spaces both notions coincide. Since we will only work with Fréchet spaces we do not have to distinguish this two notions and just write $\widehat{\otimes}$ for the completed tensor product.

1.1.15. Let A, B be K-Banach spaces with valuations ν_A, ν_B. Then there is a natural valuation $\nu_\otimes$ on $A \otimes B$ defined by

$$\nu_\otimes(a) := \sup\left\{ \inf\{\nu_A(c_j) + \nu_B(d_j) : j\} : a = \sum_j c_j \otimes d_j \right\}.$$

In this case $A\widehat{\otimes}B$ is the completion of $A \otimes B$ with respect to the valuation $\nu_\otimes$. Let $\mathcal{O}$ be the valuation ring of K and let L_A resp. L_B be $\mathcal{O}$-submodules of A resp. B. Then we have a canonical map

$$L_A \otimes_{\mathcal{O}} B \to A \otimes_K B \to A\widehat{\otimes}B.$$

which induces a semivalution on $L_A \otimes_{\mathcal{O}} L_B$. We define $L_A\widehat{\otimes}L_B$ to be the completion of $L_A \otimes_{\mathcal{O}} L_B$ with respect to this semivaluation.

Proposition 1.1.16 ([Eme11] Proposition 1.1.29)**.** *Assume that* $V = \varprojlim_n V_n$ *and* $W = \varprojlim_n W_n$ *are two Fréchet spaces and both are the projective limit of Fréchet spaces. Then there is a natural isomorphism*

$$V\widehat{\otimes}W \xrightarrow{\sim} \varprojlim_n V_n\widehat{\otimes}W_n.$$

Proposition 1.1.17 ([Eme11] Proposition 1.1.32)**.** *Let* V *and* W *be* K*-vector spaces of compact type and let* $V = \varinjlim_n V_n$ *resp.* $W = \varinjlim_n W_n$ *be expressions of* V *resp.* W *as the locally convex inductive limit of* K*-Banach spaces with injective compact transition maps. Then*

1. *There is a natural isomorphism* $\varinjlim_n V_n\widehat{\otimes}W_n \xrightarrow{\sim} V\widehat{\otimes}W$ *and thus also* $V\widehat{\otimes}W$ *is of compact type;*

2. $(V\widehat{\otimes}W)'_b \cong V'_b\widehat{\otimes}W'_b$.

1.2 Schauder bases of normed vector spaces

Definition 1.2.1. See [BGR84] 2.7.2.6. Let V be a K-Banach space with valuation ν. An orthogonal Schauder basis is a set of elements $\{v_i\}_{i\in\mathbb{N}} \subset V^{\mathbb{N}}$ fulfilling

1. $0 \le \nu(v_i) < 1$ for all $i \in \mathbb{N}$
2. For $v \in V$ there exists a unique sequence $\{c_i\}_{i\in\mathbb{N}} \in K^{\mathbb{N}}$ such that $\sum c_i v_i$ converges and $v = \sum c_i v_i$.
3. With $v = \sum c_i v_i$ as in 2. we have $\nu(v) = \inf\{\nu(c_i) + \nu(v_i) : i \in \mathbb{N}\}$.

Lemma 1.2.2. *For an index set I and $\{A_i\}_{i\in I} \in \mathbb{R}^I$, $\{B_i\}_{i\in I} \in \mathbb{R}^I$ we have the following inequalities*

$$\begin{aligned}
\inf\{A_i - B_i : i \in I\} &\le \inf\{A_i : i \in I\} - \inf\{B_i : i \in I\} \\
\inf\{A_i + B_i : i \in I\} &\ge \inf\{A_i + B_j : (i,j) \in I \times I\} \\
&= \inf\{A_i : i \in I\} + \inf\{B_i : i \in I\}
\end{aligned}$$

Proposition 1.2.3. *Let V be a K-Banach space with orthogonal Schauder basis $\{v_i : i \in \mathbb{N}\}$. Let $\{b_i \in V_b' : i \in \mathbb{N}\}$ be a set of vectors of the dual space such that $\nu(b_i(v_i)) = 0$. Assume that $\nu(b_i(v_j)) \ge 1$ for all $i \ne j$ and that*

$$\lim_{i\to\infty} \nu(b_i(v_j)) = \infty$$

for all $j \in \mathbb{N}$. Then every sequence $\{a_i\}_{i\in\mathbb{N}}$ with $a_i \in K$ and

$$\inf\{\nu(a_i) : i \in \mathbb{N}\} > -\infty$$

defines an element $\sum_i a_i b_i \in V_b'$ by

$$\left(\sum_i a_i b_i\right)\left(\sum_j c_j v_j\right) := \sum_{i,j} a_i c_j b_i(v_j).$$

Let ν_b be the valuation on V_b'. Then $\nu_b\left(\sum_i a_i b_i\right) = \inf\{\nu(a_i) + \nu_b(b_i) : i \in \mathbb{N}\}$ with $\nu_b(b_i) = -\nu(v_i)$. Moreover

$$V_b' = \left\{\sum_i a_i b_i : \inf\{\nu(a_i) : i \in \mathbb{N}\} > -\infty\right\}.$$

Proof. **Step 1:** $\sum_i a_i b_i \in V_b'$ is well defined.
Let $\sum_i c_i v_i \in V$. Because $\{c_i\}_{i\in\mathbb{N}}$ is a zero sequence, $\{a_i\}_{i\in\mathbb{N}}$ is bounded below and $\nu(b_i(v_j)) \to \infty$ for $i \to \infty$ we know that

$$\lim_{|i+j|\to\infty} a_i c_j b_i(v_j) = 0$$

and hence the sum $\sum_{i,j} a_i c_j b_i(v_j)$ converges. Using Lemma 1.2.2 we obtain

$$\begin{aligned}
\nu\left(\left(\sum_i a_i b_i\right)\left(\sum_j c_j v_j\right)\right) &= \nu\left(\sum_{i,j} a_i c_j b_i(v_j)\right) \\
&\geq \inf\{\nu(a_i) + \nu(c_j) : i, j\} \\
&= \inf\{\nu(a_i) : i\} + \inf\{\nu(c_j) : j\} \\
&\geq \inf\{\nu(a_i)\} + \nu\left(\sum_j c_j v_j\right) - 1
\end{aligned}$$

and thus the map $\sum_i a_i b_i$ is continuous.

Step 2: We will compute the valuation on V_b' in terms of the orthogonal Schauder basis $\{v_i\}_{i\in\mathbb{N}}$.

Let $\lambda \in V_b'$. Using the first inequality in Lemma 1.2.2 we obtain

$$\begin{aligned}
\nu\left(\lambda\left(\sum c_i v_i\right)\right) - \nu\left(\sum c_i v_i\right) &= \nu\left(\sum c_i \lambda(v_i)\right) - \inf\{\nu(c_i) + \nu(v_i)\} \\
&\geq \inf\{\nu(c_i) + \nu(\lambda(v_i))\} - \inf\{\nu(c_i) + \nu(v_i)\} \\
&\geq \inf\{\nu(\lambda(v_i)) - \nu(v_i)\}.
\end{aligned}$$

Thus

$$\inf\{\nu(\lambda(v_i)) - \nu(v_i)\} \geq \nu_b(\lambda) \geq \inf\{\nu(\lambda(v_i)) - \nu(v_i)\}$$

which implies $\nu_b(\lambda) = \inf\{\nu(\lambda(v_i)) - \nu(v_i)\}$.

Step 3: We will compute $\nu_b\left(\sum_i a_i b_i\right)$.

Using the description of ν_b in Step 2 and $\nu(b_i(v_j)) \geq 1$ for $i \neq j$ we can conclude that $\nu_b(b_i) = -\nu(v_i)$ and thus $\nu_b(b_i) \in (-1, 0]$. Thus by Step 2 we have

$$\begin{aligned}
\nu_b\left(\sum_i a_i b_i\right) &= \inf\left\{\nu\left(\sum_i a_i b_i(v_j)\right) - \nu(v_j) : j\right\} \\
&\geq \inf\big\{\inf\{\nu(a_i) + 1 - \delta_{i,j} : i\} - \nu(v_j) : j\big\} \\
&= \inf\big\{\inf\{\nu(a_i) + 1 - \delta_{i,j} + \nu_b(b_j) : i\} : j\big\} \\
&\geq \inf\big\{\inf\{\nu(a_i) + \nu_b(b_i) : i\} : j\big\} \\
&= \inf\big\{\nu(a_i) + \nu_b(b_i) : i\big\}
\end{aligned}$$

and hence $\nu_b\left(\sum_i a_i b_i\right) \geq \inf\{\nu(a_i) + \nu_b(b_i)\}$.

Let $J := \{j \in \mathbb{N} : \nu(a_j) = \inf\{\nu(a_i) : i \in \mathbb{N}\}\}$. For $j \in J$ and $i \neq j$ we know by the assumption on $\nu(b_i(v_j))$ that

$$\nu(a_i b_i(v_j)) \geq \nu(a_i) + 1 \geq \nu(a_j) + 1.$$

Thus $\nu(a_j b_j(v_j)) = \nu(a_j)$ implies $\nu\left(\sum_i a_i b_i(v_j)\right) = \nu(a_j)$ and hence

$$\nu\left(\sum_i a_i b_i(v_j)\right) - \nu(v_j) = \nu(a_j) - \nu(v_j) = \nu(a_j) + \nu_b(b_j).$$

Because of $\nu_b(b_i) \in (-1, 0]$ for all $i \in \mathbb{N}$ we know that

$$\inf\{\nu(a_i) + \nu_b(b_i) : i \in \mathbb{N}\} = \inf\{\nu(a_j) + \nu_b(b_j) : j \in J\}.$$

Thus we can compute

$$\begin{aligned}\nu_b\left(\sum_i a_i b_i\right) &\leq \inf\left\{\nu\left(\sum_i a_i b_i(v_j)\right) - \nu(v_j) : j \in J\right\}\\ &= \inf\{\nu(a_j) + \nu_b(b_j) : j \in J\}\\ &= \inf\{\nu(a_i) + \nu_b(b_i) : i \in \mathbb{N}\}.\end{aligned}$$

Since we already showed the opposite inequality we obtain

$$\nu_b\left(\sum_i a_i b_i\right) = \inf\{\nu(a_i) + \nu_b(b_i) : i \in \mathbb{N}\}.$$

Step 4: We will show that for every element $\lambda \in V_b'$ there exist $a_{i,1} \in K$ with $\nu(a_{i,1}) \geq \nu_b(\lambda)$ such that

$$\nu_b\left(\lambda - \sum a_{i,1} b_i\right) \geq \nu_b(\lambda) + 1.$$

For $a_{i,1} := \lambda(v_i)(b_i(v_i))^{-1}$ we know that $\nu(a_{i,1}) = \nu(\lambda(v_i)) \geq \nu_b(\lambda)$. Using $\nu(b_i(v_j)) \geq 1$ for $i \neq j$ we can compute

$$\begin{aligned}\nu\left(\left(\lambda - \sum_i a_{i,1} b_i\right)(v_j)\right) &= \nu\left(\sum_{i \neq j} a_{i,1} b_i(v_j)\right)\\ &\geq \inf\{\nu_b(\lambda) + 1 : i\}\\ &= \nu_b(\lambda) + 1\end{aligned}$$

And thus $\nu_b\left(\lambda - \sum_i a_{i,1} b_i\right) \geq 1$. Thus using induction we can find $a_{i,j} \in K$ with $\nu(a_{i,j}) \geq \nu_b(\lambda) + j - 1$ and

$$\nu_b\left(\lambda - \sum_i \left(\sum_{j=1}^n a_{i,j}\right) b_i\right) \geq \nu_b(\lambda) + n.$$

This means that $d_i := \sum_{j \geq 1} a_{i,j}$ exists and $\nu(d_i) \geq \nu_b(\lambda)$. Hence $\sum d_i b_i \in V_b'$ and $\lambda = \sum d_i b_i$. □

Lemma 1.2.4. *Let A and B be K-Banach algebras with valuations ν_A, ν_B.*

Assume that ν_A is multiplicative. Assume there exists elements $a_1, ..., a_n \in A$ and $r \in \mathbb{R}$ such that

$$\left\{\pi^{\lceil r|\mu|\rceil} a_1^{\mu_1} \cdots a_n^{\mu_n} : \mu \in \mathbb{N}_0^n\right\}$$

is an orthogonal Schauder basis of A. Assume furthermore that

$$A_b := \left\langle \pi^{\lceil r|\mu|\rceil} a_1^{\mu_1} \cdots a_n^{\mu_n} : \mu \in \mathbb{N}_0^n \right\rangle_{K-linear}$$

is K-subalgebra of A. Let

$$\overline{\varphi} : A_b \longrightarrow B$$

be a morphism of K-algebras such that $\nu_B(\overline{\varphi}(a_i)) \geq \nu_A(a_i)$ for $i \in \{1, ..., n\}$. Then there exists a unique K-Banach algebra morphism

$$\varphi : A \longrightarrow B$$

extending $\overline{\varphi}$. For an element $a = \sum a_\mu a_1^{\mu_1} \cdots a_n^{\mu_n} \in A$ we have that

$$\varphi(a) = \sum a_\mu \overline{\varphi}(a_1)^{\mu_1} \cdots \overline{\varphi}(a_n)^{\mu_n}.$$

Proof. Every $a \in A$ can be expressed as a sum

$$a = \sum_\mu a_\mu a_1^{\mu_1} \cdots b_n^{\mu_n}$$

with $\nu(a_\mu) + \sum_i \mu_i \nu_A(a_i) \to \infty$ because $\left\{\pi^{\lceil r|\mu|\rceil} a_1^{\mu_1} \cdots a_n^{\mu_n} : \mu \in \mathbb{N}_0^n\right\}$ is an orthogonal Schauder basis and ν_A is multiplicative. Since

$$\nu_B\left(\overline{\varphi}(a_1)^{\mu_1} \cdots \overline{\varphi}(a_1)^{\mu_n}\right) \geq \sum_{i=1}^n \mu_i \nu_B\left(\overline{\varphi}(a_i)\right) \geq \sum_{i=1}^n \mu_i \nu_A(a_i) = \nu_A\left(a_1^{\mu_1} \cdots a_n^{\mu_n}\right)$$

we know that

$$\lim_{|\mu| \to \infty} \nu_B\left(a_\mu \overline{\varphi}(a_1)^{\mu_1} \cdots \overline{\varphi}(a_n)^{\mu_n}\right) = \infty$$

and hence $\sum a_\mu \overline{\varphi}(a_1)^{\mu_1} \cdots \overline{\varphi}(a_n)^{\mu_n}$ converges in B. □

Lemma 1.2.5. *Let A, B, C be a K-Banach algebras with valuations ν_A, ν_B, ν_C. Assume that ν_A is multiplicative. Assume there exists elements $a_1, ..., a_n \in A$ and $r \in \mathbb{R}$ such that*

$$\left\{\pi^{\lceil |\mu| \rceil} a_1^{\mu_1} \cdots a_n^{\mu_n} : \mu \in \mathbb{N}_0^n\right\}$$

is an orthogonal Schauder basis of A. Assume furthermore that

$$A_b := \left\langle \pi^{\lceil |\mu| \rceil} a_1^{\mu_1} \cdots a_n^{\mu_n} : \mu \in \mathbb{N}_0^n \right\rangle_K$$

is an K-subalgebra of K. Consider an K-algebra morphism

$$\Delta : A_b \longrightarrow B \otimes C$$

with $\Delta(a_i) = \sum_j b_{i,j} \otimes c_{i,j}$ *and*

$$\min\{\nu_B(b_{i,j}) + \nu_C(c_{i,j}) : j\} \geq \nu_A(a_i)$$

for all $i \in \{1, ..., n\}$*. Then* Δ *can uniquely be extended to* K*-Banach algebra morphism*

$$\Delta : A \longrightarrow B\widehat{\otimes}C.$$

Proof. This follows from the definition of the valuation on $B \otimes C$ and Lemma 1.2.4. □

1.3 Locally convex K-Hopf algebras

Definition 1.3.1. A locally convex K-coalgebra is a locally convex K-Fréchet space C together with two continuous maps $\Delta : C \longrightarrow C\widehat{\otimes}C$ and $\epsilon : C \longrightarrow K$ such that the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C\widehat{\otimes}C \arrow[d, "\mathrm{id}\otimes\Delta"] \\ C\widehat{\otimes}C \arrow[r, "\Delta\otimes\mathrm{id}"'] & C\widehat{\otimes}C\widehat{\otimes}C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "\mathrm{id}"] & C\widehat{\otimes}C \arrow[d, "\mathrm{id}\otimes\epsilon"] \\ C\widehat{\otimes}C \arrow[r, "\epsilon\otimes\mathrm{id}"'] & K \otimes C \cong C \cong C \otimes K \end{tikzcd} .$$

A locally convex K-bialgebra is a locally convex Fréchet space C together with four continuous maps $(m, \eta, \Delta, \epsilon)$ such that

1. (C, m, η) is a locally convex K-algebra.
2. (C, Δ, ϵ) is a locally convex K-coalgebra.
3. Δ and ϵ are maps of locally convex K-algebras.

Equivalently to 3. one can also require m and η to be maps of locally convex K-coalgebras.

A locally convex K-Hopf algebra is a locally convex K-bialgebra $(H, m, \eta, \Delta, \epsilon)$

together with a continuous K-linear map $S : H \longrightarrow H$ such that

$$\begin{tikzcd}
 & H\widehat{\otimes}H \arrow[rr, "S\otimes\mathrm{id}"] & & H\widehat{\otimes}H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\epsilon"] & & K \arrow[rr, "\eta"] & & H \\
 & H\widehat{\otimes}H \arrow[rr, "\mathrm{id}\otimes S"'] & & H\widehat{\otimes}H \arrow[ur, "m"'] &
\end{tikzcd}$$

commutes.
S is an antihomomorphism meaning that

$$S(ab) = S(b)S(a)$$

for all $a, b \in C$.
If we replace the words "locally convex" by the word "K-Banach" in this definition we obtain the notion of a K-Banach coalgebra, a K-Banach bialgebra and a K-Banach Hopf algebra.

Definition 1.3.2. Let $(A, m_A, \eta_A, \Delta_A, \epsilon_A)$ and $(B, m_B, \eta_B, \Delta_B, \epsilon_B)$ be two locally convex K-bialgebras. A bialgebra bracket

$$\langle \cdot, \cdot \rangle : A\widehat{\otimes}B \longrightarrow K$$

is a jointly continuous K-bilinear pairing with the properties

$$\begin{aligned} \langle ab, f\rangle &= \langle a \otimes b, \Delta_B(f)\rangle; & \langle a, fg\rangle &= \langle \Delta_A(a), f \otimes g\rangle; \\ \langle a, 1\rangle &= \epsilon_A(a); & \langle 1, f\rangle &= \epsilon_B(f) \end{aligned}$$

for all $a, b \in A$ and $f, g \in B$. Here $\langle a \otimes b, \Delta_B(f)\rangle$ means $\langle a, \cdot\rangle \otimes \langle b, \cdot\rangle\,(\Delta_B(f))$ and likewise for $\langle \Delta_A(a), f \otimes g\rangle$.

Chapter 2

Quantum locally analytic functions

Locally analytic functions on a p-adic group G over K are functions

$$f : G \longrightarrow K$$

that locally can be expressed by converging power series. If G is compact, the locally analytic functions on G form a locally convex K-Hopf algebra. In this chapter we will describe a quantization of the locally convex K-Hopf algebra of locally analytic functions for the p-adic group $\mathrm{GL}(2, \mathcal{O})$, i.e. we construct a p-adic quantum group.

Since our quantum group is noncommutative, the locally convex Hopf algebra we obtain can't be interpreted as algebra of functions with values in K anymore.

Our idea is to start with the quantum matrix algebra $M_q(2, K)$, see Definition 2.1.1. There are very few K-points of $M_q(2, K)$ but we will show that for $a \in \mathcal{O}^4$ and $2 < 2r < \nu(1 - q)$ we can still make sense of the disc around a with radius r. We then will construct the K-Banach algebra of converging power series on such a disc.

This will allow us to construct a locally convex K-algebra $C_q^{la}(\mathrm{GL}(2, \mathcal{O}), K)$ which for $q = 1$ will be equal to the algebra of locally analytic functions on $\mathrm{GL}(2, \mathcal{O})$. The algebra $C_q^{la}(\mathrm{GL}(2, \mathcal{O}), K)$ is of compact type and for $q \neq 1$ it is noncommutative.

In section 2.2 we will show that $C_q^{la}(\mathrm{GL}(2, \mathcal{O}), K)$ also carries the structure of a locally convex K-Hopf algebra i.e. it is a p-adic quantum group.

2.1 Quantum locally analytic functions

In this section we will describe how to construct quantum locally analytic functions on $\mathrm{GL}(2, \mathcal{O})$. To abbreviate the notation we will write H for $\mathrm{GL}(2, \mathcal{O})$.

First we recall the definition of the algebra of quantized rational functions $M_q(2,K)$ on the space $M(2,K)$ of 2×2 matrices.

Definition 2.1.1. Let $q \in \mathcal{O}^\times$.The bialgebra $M_q(2,K)$ is defined to be the noncommutative polynomial algebra $K\{a,b,c,d\}$ modulo the following relations

$$\begin{aligned}
&ab = qba; && ac = qca; && bd = qdb;\\
&cd = qdc; && bc = cb; &&\\
&ad - da = (q-q^{-1})bc. &&&& \qquad (2.1.1)
\end{aligned}$$

The coalgebra structure is given on generators by

$$\begin{aligned}
&\Delta(a) = a\otimes a + b\otimes c; && \Delta(b) = a\otimes b + b\otimes d;\\
&\Delta(c) = c\otimes a + d\otimes c; && \Delta(d) = c\otimes b + d\otimes d;\\
&\epsilon(a) = \epsilon(d) = 1; && \epsilon(b) = \epsilon(c) = 0. \qquad (2.1.2)
\end{aligned}$$

Denote by $\det_q$ the quantum determinant $ad - qbc$. Then $\mathrm{GL}_q(2)$ is given by

$$\mathrm{GL}_q(2) := M_q(2,K)[z]/(z\det_q -1).$$

Here we used $[\cdot]$ to indicate that z is in the center of $M_q(2,K)[z]$. Note that $\det_q$ is also in the center of $M_q(2,K)[z]$. The antipode is given on generators by

$$\begin{aligned}
&S_g(a) = d\det_q^{-1}; && S_g(b) = -q^{-1}b\det_q^{-1};\\
&S_g(c) = -qc\det_q^{-1}; && S_g(d) = a\det_q^{-1}. \qquad (2.1.3)
\end{aligned}$$

2.1.2. For $r\in\mathbb{Q}$ and $z\in K$ let $B(z,r) = \{x\in K : \nu(x-z)\geq r\}$. For $g = (g_a,g_b,g_c,g_d)\in K^4 = M(2,K)$ and $r\in\mathbb{Q}$ we define the set

$$B(g,r) := B(g_a,r)\times B(g_b,r)\times B(g_c,r)\times B(g_d,r)\subseteq M(2,K).$$

Let $H := \mathrm{GL}(2,\mathcal{O})$. Recall that a function $f : H\to K$ is called locally analytic, if there is a covering $H = \coprod_i B(g_i,r_i)$ such that $f\restriction_{B(g_i,r_i)}$ is a converging power series for all i. Since H is compact we can assume that $r_i = r$ for some $r\in\mathbb{Q}$ and every i. We will denote the ring of locally analytic functions on H by $C^{la}(H,K)$.
In order to construct a quantized version of $C^{la}(H,K)$ we first have to consider what a covering of H in the quantized version should be.
For $2\leq 2r < \nu(1-q)$ and a covering $H = \coprod_i B(g_i,r)$ we will construct a ring of noncommutative power series for every $B(g_i,r)$. This will be our replacement of a covering of H.

Definition 2.1.3. For $\mu \in \mathbb{N}_0^4$ and $g = (g_a, g_b, g_c, g_d) \in K^4$ we define

$$g^\mu := (a - g_a)^{\mu_a}(b - g_b)^{\mu_b}(c - g_c)^{\mu_c}(d - g_d)^{\mu_d}.$$

Then the set $\{g^\mu\}_{\mu \in \mathbb{N}_0^4}$ is a K-basis of $M_q(2, K)$ by [Kas95] Theorem IV.4.1. For $r \in \mathbb{Q}$ we define a valuation on $M_q(2, K)$ by

$$\nu_{g,r}\left(\sum_\mu a_\mu g^\mu\right) := \min\{\nu(a_\mu) + r|\mu|\}$$

where $a_\mu \in K$ and $|\mu| = \sum \mu_i$.

2.1.4. For this chapter we will always assume that every radius r that appears is an element of $\mathbb{Q}$. Moreover we will write H for the group $\mathrm{GL}(2, \mathcal{O})$.
Now we will to show that for $2 \leq 2r < \nu(1 - q)$ the valuation $\nu_{g,r}$ is multiplicative meaning that $\nu_{g,r}(fg) = \nu_{g,r}(f) + \nu_{g,r}(g)$ for all $f, g \in M_q(2, K)$. The proof of this property needs some preparation.

2.1.5. For the next Lemma we fix some notation. Let

$$x_1 = (a - g_a), ..., x_4 = (d - g_d).$$

Then for a monomial $m = x_{i_1}x_{i_2} \cdots x_{i_n}$ with $i_j \in \{1, ..., 4\}$ we define

$$\mathrm{inv}(m) := \#\{(i_k, i_l) : k < l \text{ and } i_k > i_l\}.$$

Moreover let $\mu_i(m)$ be the number of times that x_i occurs in m and define $\mu(m) = (\mu_1(m), ..., \mu_4(m))$.

Lemma 2.1.6. *Let $\nu(1 - q) \geq 1$. Let I be the ideal in $K\{a, b, c, d\}$ that is generated by the relations (2.1.1). Let $g \in H$ and let m be a monomial in*

$$\{(a - g_a), (b - g_b), (c - g_c), (d - g_d)\} \subseteq K\{a, b, c, d\}.$$

Then there exist $a_\eta \in K$ and an $I_m \in I$ such that

$$m = g^{\mu(m)} + \sum_{|\eta| \leq |\mu(m)|} a_\eta g^\eta + I_m$$

and $\nu(a_\eta) \geq \nu(1 - q)$ for all $|\eta| = |\mu(m)|$ and $\nu(a_\eta) \geq \frac{\nu(1-q)}{2}(|\mu(m)| - |\eta|)$ for all $|\eta| < |\mu(m)|$.

Proof. Note that the relations (2.1.1) can be written as

$$\begin{aligned}
(b-g_b)(a-g_a) &= q^{-1}(a-g_a)(b-g_b) + (q^{-1}-1)g_b(a-g_a) \\
&\quad + (q^{-1}-1)g_a(b-g_b) + (q^{-1}-1)g_a g_b \\
(c-g_c)(a-g_a) &= q^{-1}(a-g_a)(c-g_c) + (q^{-1}-1)g_c(a-g_a) \\
&\quad + (q^{-1}-1)g_a(c-g_c) + (q^{-1}-1)g_a g_c \\
(c-g_c)(b-g_b) &= (b-g_b)(c-g_c) \\
(d-g_d)(c-g_c) &= q^{-1}(c-g_c)(d-g_d) + (q^{-1}-1)g_d(c-g_c) \\
&\quad + (q^{-1}-1)g_c(d-g_d) + (q^{-1}-1)g_c g_d \\
(d-g_d)(b-g_b) &= q^{-1}(b-g_b)(d-g_d) + (q^{-1}-1)g_d(b-g_b) \\
&\quad + (q^{-1}-1)g_b(d-g_d) + (q^{-1}-1)g_b g_d \\
(d-g_d)(a-g_a) &= (a-g_a)(d-g_d) \\
&\quad - (q-q^{-1})[(b-g_b)(c-g_c) + g_b(c-g_c) + g_c(b-g_b) + g_b g_c].
\end{aligned} \tag{2.1.4}$$

Sublemma: The statement in the Lemma is true for all monomials m with

$$\min\{\mu_1(m), \mu_4(m)\} = 0.$$

In order to prove the Sublemma we will use induction on $\mathrm{inv}(m)$. Let m be a monomial with $\mathrm{inv}(m) = 0$. Then $m = g^{\mu(m)}$ and hence the statement is true. Assume that $\mathrm{inv}(m) = n > 0$. Then since $\min\{\mu_1(m), \mu_4(m)\} = 0$ there exist i, j with $i > j$, $(i, j) \neq (4, 1)$ and monomials m', m'' such that $m = m' x_i x_j m''$. Using (2.1.4) we can conclude that there exists $J \in I$ and $e, f, g \in \mathcal{O}$ such that

$$\begin{aligned}
m &= m' x_j x_i m'' + (q^{-1}-1) m' x_j x_i m'' + (q^{-1}-1)(e m' x_j m'' + f m' x_i m'') \\
&\quad + (q^{-1}-1) g m' m'' + J
\end{aligned}$$

Since $\nu\left(q^{-1}-1\right) = \nu\left(q^{-1}(1-q)\right) = \nu(1-q)$ and

$$\max\left\{\mathrm{inv}(m' x_j x_i m''), \mathrm{inv}(m' x_j m''), \mathrm{inv}(m' x_i m''), \mathrm{inv}(m' m'')\right\} < \mathrm{inv}(m)$$

we can use the induction hypothesis to conclude the statement for m.

Now we show the statement of the Lemma by using induction on

$$\min\{\mu_1(m), \mu_4(m)\}.$$

The case $\min\{\mu_1(m), \mu_4(m)\} = 0$ was proven by the Sublemma. Let the statement be true for all $\bar{m}$ with $\min\{\mu_1(\bar{m}), \mu_4(\bar{m})\} < n$. Let m be a monomial with $\min\{\mu_1(m), \mu_4(m)\} = n$.

We will show the statement of the Lemma by using the induction hypothesis and using induction on $\mathrm{inv}(m)$. If $\mathrm{inv}(m) = 0$ the statement is obviously true. Let $\mathrm{inv}(m) = k$. If there exist i, j with $i > j$, $(i, j) \neq (4, 1)$ and monomials m', m'' such that $m = m' x_i x_j m''$ then we can proceed as in the proof of the Sublemma.

If not, there exist monomials m', m'' such that $m = m'(d - g_d)(a - g_a)m''$. Hence by (2.1.4) we can conclude

$$\begin{aligned} m = & m'(a - g_a)(d - g_d)m'' + \left(q - q^{-1}\right) m'(b - g_b)(c - g_c)m'' \\ & + \left(q - q^{-1}\right)\left(g_b m'(c - g_c)m'' + g_c m'(b - g_b)m'' + g_b g_c m' m''\right) + J \end{aligned}$$

for some $J \in I$. Since $\mathrm{inv}(m'(a - g_a)(d - g_d)m'') < \mathrm{inv}(m)$ we can conclude the statement for

$$m'(a - g_a)(d - g_d)m''$$

by induction hypothesis for inv. For all the other terms

$$\min\{\mu_1(\cdot), \mu_4(\cdot)\} < n$$

and thus we can conclude the statement for these terms by the induction hypothesis for $\min\{\mu_1(\cdot), \mu_4(\cdot)\}$. Since $\nu(q - q^{-1}) = \nu(1 - q)$ the Lemma is proven. □

Lemma 2.1.7. *Let $2 \leq 2r < \nu(1 - q)$ and let $g \in H$. Then the valuation $\nu_{g,r}$ on $M_q(2, K)$ defined above is multiplicative.*

Proof. By Lemma 2.1.6 we know that there exist $a_\eta \in K$ with

$$g^\mu g^{\mu'} = g^{\mu + \mu'} + \sum_{|\eta| \leq |\mu| + |\mu'|} a_\eta g^\eta$$

and $\nu\left(a_\eta\right) \geq \nu(1 - q)$ for $|\eta| = |\mu| + |\mu'|$ and

$$\nu\left(a_\eta\right) \geq \frac{\nu(1 - q)}{2}(|\mu| + |\mu'| - |\eta|)$$

for $|\eta| < |\mu| + |\mu'|$. Thus

$$\nu_{g,r}\left(a_\eta g^\eta\right) \geq \nu(1 - q) + r(|\mu| + |\mu'|) > \nu_{g,r}\left(g^{\mu + \mu'}\right)$$

for $|\eta| = |\mu| + |\mu'|$ and

$$\begin{aligned}
\nu_{g,r}\left(a_\eta g^\eta\right) &\geq \frac{\nu(1-q)}{2}(|\mu| + |\mu'| - |\eta|) + r(|\eta|) \\
&= \frac{\nu(1-q)}{2}(|\mu| + |\mu'| - |\eta|) - r(|\mu| + |\mu'| - |\eta|) + r(|\mu| + |\mu'|) \\
&= \left(\frac{\nu(1-q)}{2} - r\right)(|\mu| + |\mu'| - |\eta|) + \nu_{g,r}\left(g^{\mu+\mu'}\right) \\
&> \nu_{g,r}\left(g^{\mu+\mu'}\right)
\end{aligned}$$

for $|\eta| < |\mu| + |\mu'|$. Hence we can conclude that

$$\begin{aligned}
\nu_{g,r}\left(g^\mu g^{\mu'}\right) = \nu_{g,r}\left(g^{\mu+\mu'} + \sum_{|\eta|\leq|\mu|+|\mu'|} a_\eta g^\eta\right) &= \nu_{g,r}\left(g^{\mu+\mu'}\right) \\
&= r|\mu| + r|\mu'| = \nu_{g,r}\left(g^\mu\right) + \nu_{g,r}\left(g^{\mu'}\right).
\end{aligned}$$

Now let $P = \sum a_\mu g^\mu$ and $Q = \sum b_\mu g^\mu$. Then

$$\begin{aligned}
\nu_{g,r}(PQ) &= \nu_{g,r}\left(\sum_{\mu,\mu'} a_\mu b_{\mu'} g^\mu g^{\mu'}\right) \\
&\geq \min\left\{\nu_{g,r}\left(a_\mu b_{\mu'} g^\mu g^{\mu'}\right)\right\} \\
&= \min\left\{\nu_{g,r}\left(a_\mu g^\mu\right) + \nu_{g,r}\left(b_{\mu'} g^{\mu'}\right)\right\} \\
&= \min\left\{\nu_{g,r}\left(a_\mu g^\mu\right)\right\} + \min\left\{\nu_{g,r}\left(b_{\mu'} g^{\mu'}\right)\right\} \\
&= \nu_{g,r}(P) + \nu_{g,r}(Q)
\end{aligned}$$

implies that $\nu_{g,r}$ is submultiplicative. To show that $\nu_{g,r}$ is indeed multiplicative we can use the same proof as in the commutative case which we will shortly recall. We may assume that $r \in \mathbb{Z}$. Otherwise we could just enlarge the field and rescale the valuation. Let $I \subseteq \mathcal{O}\left\{\frac{a-g_a}{\pi^r}, \frac{b-g_b}{\pi^r}, \frac{c-g_c}{\pi^r}, \frac{d-g_d}{\pi^r}\right\}$ be the ideal generated by the relations (2.1.1). Note that the relations (2.1.1) generate an ideal in $\mathcal{O}\left\{\frac{a-g_a}{\pi^r}, \frac{b-g_b}{\pi^r}, \frac{c-g_c}{\pi^r}, \frac{d-g_d}{\pi^r}\right\}$ since $\nu(1-q) > 2r$ and $\nu(q-q^{-1}) > 2r$. Then

$$A := \{f \in M_q(2,K) : \nu_{g,r}(f) \geq 0\} = \mathcal{O}\left\{\frac{a-g_a}{\pi^r}, \frac{b-g_b}{\pi^r}, \frac{c-g_c}{\pi^r}, \frac{d-g_d}{\pi^r}\right\}/I.$$

Let $\kappa := \mathcal{O}/(\pi)$ be the residue field of $\mathcal{O}$. The map

$$\mathcal{O}\left\{\frac{a-g_a}{\pi^r}, \frac{b-g_b}{\pi^r}, \frac{c-g_c}{\pi^r}, \frac{d-g_d}{\pi^r}\right\} \to \kappa[\alpha, \beta, \gamma, \delta]$$

extending $\mathcal{O} \to \kappa$ and sending $\frac{a-g_a}{\pi^r}$ to α etc. factors through A since the equations (2.1.4) are in its kernel. An element $f \in A$ is send to zero iff

$\nu_{g,r}(f) > 0$. Since $\kappa[\alpha, \beta, \gamma, \delta]$ is an integral domain it follows that $\nu_{g,r}$ is multiplicative. $\square$

Remark 2.1.8. From the relations (2.1.4) we see that if $2r > \nu(1-q)$ and $g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ the valuation $\nu_{g,r}$ on $M_q(2, K)$ is not submultiplicative and hence its completion is not an algebra anymore. Thus in the quantized case there do not exist arbitrarily small neighborhoods of elements $g \in H$.

Definition 2.1.9. Let $g \in H$ and let $2 \leq 2r < \nu(1-q)$. We define the quantum analytic functions $C_q^{an}(g, r)$ on the disc $B(g, r)$ to be the completion of $M_q(2, K)$ with respect to the valuation $\nu_{g,r}$. As a vector space we can describe $C_q^{an}(g, r)$ as

$$C_q^{an}(g, r) = \left\{ \sum_{\mu \in \mathbb{N}_0^4} a_\mu g^\mu : \nu(a_\mu) + r|\mu| \to \infty \right\}.$$

Lemma 2.1.10. *$C_q^{an}(g, r)$ together with the valuation $\nu_{g,r}$ is a Banach algebra. The valuation $\nu_{\beta,r}$ is the same for every element $\beta \in B(g, r)$. Hence also $C_q^{an}(\beta, r)$ is the same K-Banach algebra for all $\beta \in B(g, r)$.*

Proof. Since $C_q^{an}(g, r)$ is the completion of a K-algebra by a multiplicative valuation it is a K-Banach algebra. The proof of the base point independence is very similar as in the commutative case, see e.g. [Sch06] Theorem 25.1. $\square$

Lemma 2.1.11. *Let $g \in H$ and let $2 \leq 2r \leq 2r' \leq \nu(1-q)$. For $g' \in B(g, r)$ the algebra morphism $C_q^{an}(g, r) \longrightarrow C_q^{an}(g', r')$ induced by the identity map on $M_q(2, K)$ is continuous, injective and compact.*

Proof. We can assume that $g = g'$ by Lemma 2.1.10. Thus Lemma 1.1.10 implies $C_q^{an}(g, r) \longrightarrow C_q^{an}(g', r')$ is continuous, injective and compact. $\square$

Remark 2.1.12. Let $r \geq 1$. There exists a finite set I and $\{(g_i)_{i \in I} : g_i \in H\}$ such that as topological spaces we have

$$H = \coprod_{i \in I} B(g_i, r).$$

Definition 2.1.13. Let $2 \leq 2r < \nu(1-q)$ and let $H = \coprod_{i \in I} B(g_i, r)$. Then we define the algebra

$$C_q^{la}(H, r) := \prod_{i \in I} C_q^{an}(g_i, r)$$

where the multiplication is componentwise. We will denote by ν_r the usual product valuation defined on $f = \prod_i f_i$ by

$$\nu_r(f) = \min\{\nu_{g_i,r}(f_i) : i \in I\}.$$

Then $C_q^{la}(H,r)$ together with ν_r is a K-Banach algebra. By Lemma 2.1.10 this definition is independent of choices.

Lemma 2.1.14. *Let $2 \leq 2r < 2r' < \nu(1-q)$ and let $B(g,r) = \coprod_i B(g_i,r')$. Then the K-Banach algebra morphism*

$$\begin{aligned} C_q^{an}(g,r) &\longrightarrow \textstyle\prod_i C_q^{an}(g_i,r') \\ f &\longmapsto \textstyle\prod_i f \end{aligned}$$

is continuous, injective and compact.
Let $H = \coprod_i B(g_i,r)$ and let $B(g_i,r) = \coprod_j B(g_{i_j},r')$. Then the K-Banach algebra morphism

$$\begin{aligned} C_q^{la}(H,r) &\longrightarrow C_q^{la}(H,r') \\ \textstyle\prod_i f_i &\longmapsto \textstyle\prod_i \left(\prod_j f_i\right) \end{aligned}$$

is continuous, injective and compact. Both maps are independent of choices by Lemma 2.1.10.

Proof. This is a direct consequence of Lemma 1.1.11. □

Definition 2.1.15. We define the K-algebra of quantum locally analytic functions on H by

$$C_q^{la}(H,K) := \varinjlim_{2r<\nu(1-q)} C_q^{la}(H,r).$$

For $q = 1$ we obtain the usual algebra of locally analytic functions on H.

Proposition 2.1.16. *The space of quantum locally analytic functions $C_q^{la}(H,K)$ is of compact type. Moreover it is reflexive, bornological and complete and its strong dual is a Fréchet space.*

Proof. The first claim follows from Lemma 2.1.14. The other claims then follow from [Sch02] Proposition 16.10. □

Lemma 2.1.17. *For $r, s \in \mathbb{R}$ and $2 \leq 2r < \nu(1-q)$ let*

$$L_{r,s} := \{f \in C_q^{la}(H,r) : \nu_r(f) \geq s\}.$$

Let $\{r_n\}_{n\in\mathbb{N}} \in \mathbb{Q}_{\geq 1}^{\mathbb{N}}$ be a strictly monotonously increasing sequence converging to $\frac{\nu(1-q)}{2}$. Then the family of lattices

$$\left\{ \sum_n i_{r_n}\left(L_{r_n,j(r_n)}\right) : j(r_n) \in \mathbb{Q} \right\}$$

defines the topology on $C_q^{la}(H,K)$.

Proof. This is [Sch02] Lemma 5.1, see also 1.1.5. □

Lemma 2.1.18. *For $2 \leq 2r < \nu(1-q)$ and $g \in H$ we have that*

$$det_q := ad - qbc$$

is invertible in $C_q^{an}(g,r)$ and $\nu_{g,r}(\det_q) = 0$. Hence it is also invertible in $C_q^{la}(H,r)$ and $C_q^{la}(H,K)$. Thus $\mathrm{GL}_q(2)$ is a subalgebra of $C_q^{an}(g,r)$, $C_q^{la}(H,r)$ and $C_q^{la}(H,K)$.

Proof. Let $\det_q(g) := g_a g_d - q g_b g_c$ and let

$$\begin{aligned}
\alpha &:= (a-g_a)(d-g_d) + (a-g_a)g_d + (d-g_d)g_a \\
\beta &:= (b-g_b)(c-g_c) + (b-g_b)g_c + (c-g_c)g_b.
\end{aligned}$$

Then

$$\begin{aligned}
\det_q &= \alpha - q\beta + \det_q(g) \\
&= \det_q(g)\left(1 - \frac{-\alpha + q\beta}{\det_q(g)}\right).
\end{aligned}$$

Because of $\nu(1-q) \geq 2$ and $g \in \mathrm{GL}(2,\mathcal{O})$ we know that $\det_q(g) \in \mathcal{O}^\times$. Furthermore

$$\nu_{g,r}\left(-\alpha + q\beta\right) = r > 0$$

implies that $\det_q$ is invertible by using the geometric series. $\square$

2.2 $C_q^{la}(H,K)$ as a locally convex K-Hopf algebra

In order to be able to speak of a p-adic quantum group we need to have some locally convex K-Hopf algebra structure on the quantum locally analytic functions. In this section we will construct the coproduct, the counit and the antipode for $C_q^{la}(H,K)$ by extending the corresponding maps already known for $\mathrm{GL}_q(2)$.

2.2.1. As noted in 1.1.14 there are two common notions of the completed tensor product $\widehat{\otimes}_K$. But since $C_q^{la}(H,r)$ is a Banach space and $C_q^{la}(H,K)$ fulfills the requirements of [Eme11] Proposition 1.1.31, both notions coincide in our case. We will omit the K and just write $\widehat{\otimes}$ for $\widehat{\otimes}_K$.
For $2 \leq 2r < \nu(1-q)$ and $g \in H$ we will denote by

$$i_{g,r} : C_q^{an}(g,r) \longrightarrow C_q^{la}(H,r)$$

the natural inclusion map. Recall from **1.1.15** that for two K-Banach spaces A, B we defined a valuation on $A\widehat{\otimes}B$.

Proposition 2.2.2. *Let $g \in H$ and $2 \leq 2r < \nu(1-q)$. Denote by Δ the comultiplication map $\Delta : M_q(2,K) \longrightarrow M_q(2,K) \otimes M_q(2,K)$. Then*

$$f \mapsto \sum_{B(\alpha,r)\cdot B(\beta,r) \subseteq B(g,r)} i_{\alpha,r} \otimes i_{\beta,r}(\Delta(f))$$

defines an algebra morphism

$$M_q(2,K) \to C_q^{la}(H,r) \otimes C_q^{la}(H,r)$$

which can uniquely be extended to a continuous K-Banach algebra morphism

$$\Delta_{g,r} : C_q^{an}(g,r) \longrightarrow C_q^{la}(H,r) \widehat{\otimes} C_q^{la}(H,r).$$

The morphism $\Delta_{g,r}$ is valuation increasing.

Proof. First we check that the map $M_q(2,K) \longrightarrow C_q^{la}(H,r) \otimes C_q^{la}(H,r)$ is an algebra morphism. We will use that $\Delta : M_q(2,K) \to M_q(2,K) \otimes M_q(2,K)$ is an algebra morphism. Let $f, g \in M_q(2,K)$. Note that $(\alpha', \beta') \notin B(\alpha,r) \times B(\beta,r)$ implies

$$\Big[i_{\alpha,r} \otimes i_{\beta,r}(\Delta(f))\Big] \cdot \Big[i_{\alpha',r} \otimes i_{\beta',r}(\Delta(f))\Big] = 0.$$

Thus we can compute

$$\sum_{B(\alpha,r)B(\beta,r) \subseteq B(g,r)} i_{\alpha,r} \otimes i_{\beta,r}(\Delta(fg)) = \sum_{B(\alpha,r)B(\beta,r) \subseteq B(g,r)} i_{\alpha,r} \otimes i_{\beta,r}(\Delta(f)\Delta(g))$$
$$= \sum_{B(\alpha,r)B(\beta,r) \subseteq B(g,r)} \big[i_{\alpha,r} \otimes i_{\beta,r}(\Delta(f))\big] \cdot \big[i_{\alpha,r} \otimes i_{\beta,r}(\Delta(g))\big]$$
$$= \left(\sum_{B(\alpha,r)B(\beta,r) \subseteq B(g,r)} i_{\alpha,r} \otimes i_{\beta,r}(\Delta(f)) \right) \cdot \left(\sum_{B(\alpha,r)B(\beta,r) \subseteq B(g,r)} i_{\alpha,r} \otimes i_{\beta,r}(\Delta(g)) \right).$$

The set $\left\{\pi^{\lceil r|\mu| \rceil} g^\mu : \mu \in \mathbb{N}_0^4\right\}$ is an orthogonal Schauder basis of $C_q^{an}(g,r)$. By Lemma 1.2.5 it suffices to show that for $x \in \{a,b,c,d\}$ and

$$\Delta(x - g_x) = \sum_j z_{x,j} \otimes w_{x,j}$$

we have that

$$\min\left\{\nu_{g,r}(z_{x,j}) + \nu_{g,r}(w_{x,j}) : j\right\} \geq r = \nu_{g,r}(x - g_x).$$

Therefore consider for $\alpha, \beta \in H$ with $\alpha \cdot \beta \in B(g,r)$ the formulas

$$\begin{aligned}
\Delta\,(a-g_a) &= a\otimes a + b\otimes c - g_a \\
&= (a-\alpha_a)\otimes(a-\beta_a) + \alpha_a\otimes(a-\beta_a) + (a-\alpha_a)\otimes\beta_a \\
&+ (b-\alpha_b)\otimes(c-\beta_c) + \alpha_b\otimes(c-\beta_c) + (b-\alpha_b)\otimes\beta_c \\
&+ (\alpha_a\beta_a + \alpha_b\beta_c - g_a) \qquad (2.2.1)
\end{aligned}$$

$$\begin{aligned}
\Delta(b-g_b) &= a\otimes b + b\otimes d - g_b \\
&= (a-\alpha_a)\otimes(b-\beta_b) + \alpha_a\otimes(b-\beta_b) + (a-\alpha_a)\otimes\beta_b \\
&+ (b-\alpha_b)\otimes(d-\beta_d) + \alpha_b\otimes(d-\beta_d) + (b-\alpha_b)\otimes\beta_d \\
&+ (\alpha_a\beta_b + \alpha_b\beta_d - g_b)
\end{aligned}$$

$$\begin{aligned}
\Delta(c-g_c) &= (c-\alpha_c)\otimes(a-\beta_a) + \alpha_c\otimes(a-\beta_a) + (c-\alpha_c)\otimes\beta_a \\
&+ (d-\alpha_d)\otimes(c-\beta_c) + \alpha_d\otimes(c-\beta_c) + (d-\alpha_d)\otimes\beta_c \\
&+ (\alpha_c\beta_a + \alpha_d\beta_c - g_c)
\end{aligned}$$

$$\begin{aligned}
\Delta(d-g_d) &= (c-\alpha_c)\otimes(b-\beta_b) + \alpha_c\otimes(b-\beta_b) + (c-\alpha_c)\otimes\beta_b \\
&+ (d-\alpha_d)\otimes(d-\beta_d) + \alpha_d\otimes(d-\beta_d) + (d-\alpha_d)\otimes\beta_d \\
&+ (\alpha_c\beta_b + \alpha_d\beta_d - g_d).
\end{aligned}$$

Since $\alpha\beta \in B(g,r)$ we can conclude that $\nu(\alpha_a\beta_a + \alpha_b\beta_c - g_a) \geq r$. Moreover $\alpha_x, \beta_x \in \mathcal{O}$ for $x \in \{a,b,c,d\}$ and $\alpha, \beta \in H$ and hence e.g.

$$\nu_{g,r}(\alpha_x \otimes (y-\beta_y)) \geq r$$

for $x, y \in \{a,b,c,d\}$. There are similar estimates for the other terms of the right hand side of equation (2.2.1). Thus we can conclude for

$$i_{\alpha,r}\otimes i_{\beta,r}\,(\Delta\,(a-g_a)) = \sum_j z_{a,j}\otimes w_{a,j}$$

with $z_{a,j}, w_{a,j}$ obtained from equation (2.2.1), that

$$\min\{\nu_{g,r}(z_{a,j}) + \nu_{g,r}(w_{a,j}) : j\} \geq r = \nu_{g,r}(a-g_a).$$

Using the same argument for $x \in \{b,c,d\}$, Lemma 1.2.5 implies that we can extend Δ uniquely to a K-Banach algebra morphism

$$\Delta_{g,r} : C_q^{an}(g,r) \longrightarrow C_q^{la}(H,r)\widehat{\otimes}C_q^{la}(H,r)$$

which is valuation increasing. $\square$

Corollary 2.2.3. *Let $H = \coprod_i B(g_i, r)$ with $2 \leq 2r < \nu(1-q)$. Then the map*

$$\begin{aligned} \Delta_r : C_q^{la}(H,r) &\longrightarrow C_q^{la}(H,r)\widehat{\otimes} C_q^{la}(H,r) \\ \textstyle\prod_i f_i &\longmapsto \textstyle\sum_i \Delta_{g_i,r}(f_i). \end{aligned} \tag{2.2.2}$$

is a continuous morphism of K-Banach algebras which is valuation increasing.

Proposition 2.2.4. *The map Δ_r induces a continuous K-algebra morphism*

$$\Delta_r : C_q^{la}(H,r) \longrightarrow C_q^{la}(H,K)\widehat{\otimes} C_q^{la}(H,K).$$

Proof. By Lemma 2.1.16 $C_q^{la}(H,K)$ is of compact type. Thus by [Eme11] Proposition 1.1.32 (see Proposition 1.1.17) there is a natural isomorphism

$$\varinjlim_{2r<\nu(1-q)} C_q^{la}(H,r)\widehat{\otimes} C_q^{la}(H,r) \xrightarrow{\sim} C_q^{la}(H,K)\widehat{\otimes} C_q^{la}(H,K).$$

and thus there is a continuous map

$$\Delta_r : C_q^{la}(H,r) \longrightarrow C_q^{la}(H,r)\widehat{\otimes} C_q^{la}(H,r) \longrightarrow C_q^{la}(H,K)\widehat{\otimes} C_q^{la}(H,K).$$

$\square$

Lemma 2.2.5. *Let $g \in H$ and let $B(g,l) = \coprod_i B(g_i, l+1)$ for $l \in \mathbb{N}$ and $2(l+1) < \nu(1-q)$. Then the diagram*

$$\begin{tikzcd} C_q^{an}(g,l) \arrow[rr, "\Delta_l\circ\Delta_{g,l}"] \arrow[d, "f\mapsto\prod_i f"'] & & C_q^{la}(H,K)\widehat{\otimes} C_q^{la}(H,K) \\ \prod_i C_q^{an}(g_i,l+1) \arrow[urr, "\sum_i \Delta_{l+1}\circ\Delta_{g_i,l+1}"'] & & \end{tikzcd}$$

commutes.

Proof. Since all maps are continuous, it is enough to check the statement on the subalgebra $M_q(2,K)$ which is dense in $C_q^{an}(g,l)$. Let $f \in M_q(2,K)$. Then

$$\Delta_{g,l}(f) = \sum_{B(\alpha,l)B(\beta,l)\subseteq B(g,l)} i_{\alpha,l} \otimes i_{\beta,l}(\Delta(f))$$

and

$$\begin{aligned}
\sum_i \Delta_{g_i,l+1}(f) &= \sum_i \sum_{B(\gamma,l+1)B(\delta,l+1)\subseteq B(g_i,l+1)} i_{\gamma,l+1} \otimes i_{\delta,l+1}(\Delta(f)) \\
&= \sum_{B(\gamma,l+1)B(\delta,l+1)\subseteq B(g,l)} i_{\gamma,l+1} \otimes i_{\delta,l+1}(\Delta(f)) \\
&= \sum_{B(\alpha,l)B(\beta,l)\subseteq B(g,l)} \sum_{B(\gamma,l+1)\subseteq B(\alpha,l)} \sum_{B(\delta,l+1)\subseteq B(\beta,l)} i_{\gamma,l+1} \otimes i_{\delta,l+1}(\Delta(f)) \\
&= \sum_{B(\alpha,l)B(\beta,l)\subseteq B(g,l)} \sum_{B(\gamma,l+1)\subseteq B(\alpha,l)} i_{\gamma,l+1} \otimes i_{\beta,l}(\Delta(f)) \\
&= \sum_{B(\alpha,l)B(\beta,l)\subseteq B(g,l)} i_{\alpha,l} \otimes i_{\beta,l}(\Delta(f)) \\
&= \Delta_{g,l}(f).
\end{aligned}$$

□

Corollary 2.2.6. *For $r, r' \in \mathbb{Q}$ with $2 \leq 2r \leq 2r' < \nu(1-q)$ the diagram*

$$\begin{tikzcd}
C_q^{an}(H,r) \arrow[rr, "\Delta_r"] \arrow[d] & & C_q^{la}(H,K)\widehat{\otimes}C_q^{la}(H,K) \\
C_q^{an}(H,r') \arrow[urr, "\Delta_{r'}"'] & &
\end{tikzcd}$$

commutes.

Proof. If $\lceil r \rceil = \lceil r' \rceil$ then we can use the same covering of H for the definition of Δ_r and $\Delta_{r'}$ and the claim is immediate. If $\lceil r \rceil = \lceil r' \rceil - 1$ then we can use the same proof as in the previous Lemma. To obtain the statement of the corollary just combine the two cases as often as needed. □

Corollary 2.2.7. *The map*

$$\Delta := \varinjlim_{2r<\nu(1-q)} \Delta_r$$

defines a continuous morphism of locally convex K-algebras of compact type

$$\Delta : C_q^{la}(H,K) \longrightarrow C_q^{la}(H,K)\widehat{\otimes}C_q^{la}(H,K).$$

Proof. The existence of the continuous K-algebra morphism follows from the previous corollary. By Lemma 2.1.16 $C_q^{la}(H,K)$ is of compact type and thus by [Eme11] Proposition 1.1.32 (see Proposition 1.1.17) also $C_q^{la}(H,K)\widehat{\otimes}C_q^{la}(H,K)$ is of compact type. □

Lemma 2.2.8. *The counit ε on $M_q(2,K)$ given by*

$$\varepsilon(a) = \varepsilon(d) = 1; \qquad\qquad \varepsilon(b) = \varepsilon(c) = 0$$

can be extended to a K-Banach algebra morphism

$$\varepsilon_{e,r} : C_q^{la}(e,r) \longrightarrow K$$

where e is the identity element in H.

Proof. According to Lemma 1.2.4 we only have to show that $\nu(\epsilon(x - e_x)) \geq r$ for $x \in \{a, b, c, d\}$. But since $\epsilon(x - e_x) = 0$ for $x \in \{a, b, c, d\}$ this is obvious. $\square$

Lemma 2.2.9. *We have K-Banach algebra maps*

$$\varepsilon_r : C_q^{la}(H,r) \longrightarrow K$$

given by $\varepsilon_r = \varepsilon_{e,r} \circ \mathrm{pr}_e$ that are compatible for distinct r. Here pr_e is the natural projection $C_q^{la}(H,r) \longrightarrow C_q^{an}(e,r)$.

Definition 2.2.10. We define

$$\varepsilon := \varinjlim_{2r < \nu(1-q)} \varepsilon_r,$$

which is a continuous K-algebra map. There is a canonical inclusion map $\eta_{g,r} : K \to M_q(2,K) \to C_q^{an}(g,r)$. Then we can define $\eta_r : K \to C_q^{la}(H,r)$ by sending $a \in K$ to $\prod_i \eta_{g_i,r}(a)$ and we obtain a continuous K-algebra morphism

$$\eta : K \longrightarrow C_q^{la}(H,K).$$

Lemma 2.2.11. *The antihomomorphism $S_{g,r} : M_q(2,K) \longrightarrow C_q^{an}(g^{-1},r)$ defined by*

$$S_{g,r}(a) = d \det_q^{-1}; \qquad S_{g,r}(b) = -q^{-1} b \det_q^{-1};$$
$$S_{g,r}(c) = -qc \det_q^{-1}; \qquad S_{g,r}(d) = a \det_q^{-1}.$$

uniquely extends to a continuous antiautomorphism of K-Banach algebras

$$S_{g,r} : C_q^{an}(g,r) \longrightarrow C_q^{an}(g^{-1},r).$$

Proof. That the map $S_{g,r} : M_q(2,K) \longrightarrow C_q(g^{-1},r)$ is an antihomomorphism follows from the fact that $S : M_q(2,K) \longrightarrow M_q(2,K)$ is an antihomomorphism, see e.g. [Kas95] Theorem III.3.4 and Theorem IV.6.1.
Using Lemma 1.2.4 we see that in order to extend S, we only have to show $\nu_{g^{-1},r}\left(S_{g,r}(x - g_x)\right) \geq r$ for elements $x \in \{a, b, c, d\}$. We will only show this

for $x = a$ as the other cases are similar.
Recall that $\det_q(g) := g_a g_d - q g_b g_c$. Hence

$$\det_q(g^{-1}) = \det(g^{-1}) - (q-1)\left(g^{-1}\right)_b \left(g^{-1}\right)_c.$$

Thus

$$\left(\det_q(g^{-1})\right)^{-1} = \det(g) + o \tag{2.2.3}$$

where $\nu(o) \geq \nu(1-q)$. By the proof of Lemma 2.1.18

$$\det_q^{-1} = \left(\det_q(g^{-1})\right)^{-1}(1+f) \tag{2.2.4}$$

for some $f \in C_q^{an}(g^{-1}, r)$ with $\nu_{g^{-1},r}(f) \geq r$. Using (2.2.3) and (2.2.4) we obtain $\det_q^{-1}(\det(g))^{-1} = 1 + v$ for some $v \in C_q^{an}(g^{-1}, r)$ with $\nu_{g^{-1},r}(v) \geq r$. Therefore

$$\nu_{g^{-1},r}\left(\det_q^{-1}(\det(g))^{-1} - 1\right) \geq r.$$

By using $\left(g^{-1}\right)_d = g_a \det(g)^{-1}$ we can conclude

$$\begin{aligned}
S_{g,r}(a - g_a) &= d \det_q^{-1} - g_a \\
&= \det_q^{-1}\left(d - \left(g^{-1}\right)_d\right) + \left(g^{-1}\right)_d \det_q^{-1} - g_a \\
&= \det_q^{-1}\left(d - \left(g^{-1}\right)_d\right) + g_a\left(\det_q^{-1}(\det(g))^{-1} - 1\right).
\end{aligned}$$

Since $\nu_{g^{-1},r}\left(\det_q^{-1}\right) = 0$ by Lemma 2.1.18 this implies $\nu_{g^{-1},r}\left(S_{g,r}(a - g_a)\right) \geq r$. Thus Lemma 1.2.4 implies that we obtain a continuous antihomomorphism

$$S_{g,r} : C_q^{an}(g, r) \longrightarrow C_q^{an}(g^{-1}, r).$$

Next we will show that $S_{g,r}$ is an antiautomorphism. By [KS97] Chapter 9.2 Proposition 10 the following equations hold in $\mathrm{GL}_q(2)$.

$$\begin{aligned}
S_{g,r} \circ S_{g^{-1},r}(a) &= S_{g^{-1},r} \circ S_{g,r}(a) = a; \\
S_{g,r} \circ S_{g^{-1},r}(b) &= S_{g^{-1},r} \circ S_{g,r}(b) = q^{-2}b; \\
S_{g,r} \circ S_{g^{-1},r}(c) &= S_{g^{-1},r} \circ S_{g,r}(c) = q^{2}c; \\
S_{g,r} \circ S_{g^{-1},r}(d) &= S_{g^{-1},r} \circ S_{g,r}(d) = d.
\end{aligned}$$

Now consider the algebra morphism $\psi : M_q(2, K) \longrightarrow C_q^{an}(g, r)$ defined by

$$\psi(a) = a; \qquad \psi(b) = q^2 b;$$
$$\psi(c) = q^{-2}c; \qquad \psi(d) = d.$$

Then $\nu_{g,r}(\psi(x - g_x)) \geq r = \nu_{g,r}(x - g_x)$ for $x \in \{a, b, c, d\}$. We will show this for $x = b$. Consider

$$\psi(b - g_b) = q^{-2}b - g_b = q^{-2}(g - g_b) + \left(q^{-2} - 1\right) g_b$$

Now since

$$\nu_{g,r}\left(\left(q^2 - 1\right) g_b\right) = \nu\left(\left(q^{-1} + 1\right)\left(q^{-1} - 1\right) g_b\right) \geq \nu(1 - q) > r$$

the claim follows. Thus Lemma 1.2.4 implies that ψ extends to a continuous algebra morphism

$$\psi : C_q^{an}(g, r) \longrightarrow C_q^{an}(g, r)$$

which is the inverse of $S_{g^{-1},r} \circ S_{g,r}$. With the same argument one shows that $S_{g,r} \circ S_{g^{-1},r}$ is invertible and thus also $S_{g,r}$ is invertible. $\square$

Definition 2.2.12. Let $H = \coprod_i B(g_i, r)$ and let $2 \leq 2r < \nu(1 - q)$. Then we define the continuous antiautomorphism

$$S_r : C_q^{la}(H, r) \longrightarrow C_q^{la}(H, r)$$

by

$$S_r = \prod_i S_{g_i,r}.$$

Moreover we define $S : C_q^{la}(H, K) \longrightarrow C_q^{la}(H, K)$ to be

$$S := \varinjlim_{2r<\nu(1-q)} S_r.$$

Lemma 2.2.13. *$S_{e,r}$, S_r and S are antipodes.*

Proof. Let $m_r : C_q^{la}(H, r)\widehat{\otimes}C_q^{la}(H, r) \longrightarrow C_q^{la}(H, r)$ be the multiplication map. We have to show that

$$m_r \circ (S \otimes \mathrm{id}) \circ \Delta_r = \eta_r \circ \epsilon_r = m_r \circ (\mathrm{id} \otimes S) \circ \Delta_r. \tag{2.2.5}$$

For $g, g' \in H$ with $B(g, r) \neq B(g', r)$ we have that the map

$$C_q^{an}(g, r)\widehat{\otimes}C_q^{an}(g', r) \longrightarrow C_q^{la}(H, r)\widehat{\otimes}C_q^{la}(H, r) \xrightarrow{m_r} C_q^{la}(H, r)$$

is the zero map. Thus using that $S_{g,r}$ is a map from $C_q^{an}(g, r)$ to $C_q^{an}(g^{-1}, r)$ we only have to consider $\Delta_{e,r}$. It is known (see e.g. [KS97] chapter 9.2.3) that

$$\begin{tikzcd}
\mathrm{GL}_q(2) \arrow[r, "\Delta"] \arrow[d, "\eta_r\circ\epsilon_r"] & \mathrm{GL}_q(2) \otimes \mathrm{GL}_q(2) \arrow[d, "\mathrm{id}\otimes S_r"] \\
\mathrm{GL}_q(2) & \mathrm{GL}_q(2) \otimes \mathrm{GL}_q(2) \arrow[l, "m_r"]
\end{tikzcd}$$

commutes. Since $\mathrm{GL}_q(2)$ is dense in $C_q^{an}(e,r)$ we conclude that also

$$\begin{array}{ccc} C_q^{an}(e,r) & \xrightarrow{\mathrm{pr}_{g\times g^{-1}}\circ\Delta_{e,r}} & C_q^{an}(g,r)\widehat{\otimes}C_q^{an}(g^{-1},r) \\ \Big\downarrow{\scriptstyle \mathrm{pr}_g\circ\eta_r\circ\epsilon_r} & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes S_r} \\ C_q^{an}(g,r) & \xleftarrow{m_r} & C_q^{an}(g,r)\widehat{\otimes}C_q^{an}(g,r) \end{array}$$

commutes for any $g\in H$. This implies the first equality in (2.2.5). The second equality is obtained in the same way. Thus S_r and $S_{e,r}$ are antipodes. Since $S=\varinjlim S_r$ we can conclude that also S is an antipode. □

Lemma 2.2.14. *The maps $\Delta_{e,r}$, Δ_r and Δ are comultiplication morphisms.*

Proof. As in the proof of Lemma 2.2.13 this follows from the fact that

$$\Delta : M_q(2,K)\to M_q(2,K)\otimes M_q(2,K)$$

is a comultiplication map. □

Theorem 2.2.15. *Let $2\le 2r<\nu(1-q)$. Then*

1. $\left(C_q^{an}(e,r),m_{e,r},\eta_{e,r},\Delta_{e,r},\epsilon_{e,r},S_{e,r}\right)$ *is a K-Banach Hopf algebra.*
2. $\left(C_q^{la}(H,r),m_r,\eta_r,\Delta_r,\epsilon_r,S_r\right)$ *is a K-Banach Hopf algebra.*
3. $\left(C_q^{la}(H,K),m,\eta,\Delta,\epsilon,S\right)$ *is a complete locally convex K-Hopf algebra of compact type.*

Proof. Since we already showed the necessary properties for the respective maps this is now immediate. □

2.2.16. By Theorem 2.2.15 we have constructed our p-adic quantum group

$$\left(C_q^{la}(H,K),m,\eta,\Delta,\epsilon,S\right),$$

which is a deformation of $C^{la}(H,K)$ and which is equal to $C^{la}(H,K)$ if $q=1$.

2.3 Some t-adic quantum algebras

Because we need t-adic algebras in the next chapter, we will introduce some notation concerning the t-adic quantum matrix algebra and some subalgebras.

Definition 2.3.1. Let $K[t]\{a,b,c,d\}$ be the noncommutative polynomial ring in the variables $\{a,b,c,d\}$ over $K[t]$. Let $K[[t]]\langle a,b,c,d\rangle$ be its t-adic completion. Then $q:=e^t=\sum_{k\ge 0}\frac{t^k}{k!}\in K[[t]]\langle a,b,c,d\rangle$. We define $M_q(2,t)$ to be the quotient of $K[[t]]\langle a,b,c,d\rangle$ by the relations

$$\begin{aligned}
&ab = qba; && ac = qca; && bd = qdb; \\
&cd = qdc; && bc = cb; && \\
&ad - da = (q - q^{-1})bc. && &&
\end{aligned} \tag{2.3.1}$$

Then $M_q(2,t)$ is a topological bialgebra with coalgebra structure given on topological generators by

$$\begin{aligned}
\Delta(a) &= a \otimes a + b \otimes c, & \Delta(b) &= a \otimes b + b \otimes d \\
\Delta(c) &= c \otimes a + d \otimes c, & \Delta(d) &= c \otimes b + d \otimes d \\
\epsilon(a) &= \epsilon(d) = 1 & \epsilon(b) &= \epsilon(c) = 0.
\end{aligned}$$

Definition 2.3.2. Let $r_t > \frac{2e}{p-1}$ and let

$$K[[t]]_{r_t} := \left\{ \sum_{i \geq 0} a_i t^i : a_i \in K \text{ and } \lim_{i \to \infty} (\nu(a_i) + i r_t) = \infty \right\} \subseteq K[[t]]$$

On $K[[t]]_{r_t}$ we can define a multiplicative valuation ν by setting

$$\nu\left(\sum_i a_i t^i\right) = \min\{\nu(a_i) + i r_t\}.$$

This valuation extends the valuation on K. We define

$$K[[t]]^\circ_{r_t} := \{f \in K[[t]]_{r_t} : \nu(f) \geq 0\}.$$

Note that both $K[[t]]_{r_t}$ and $K[[t]]^\circ_{r_t}$ are complete K-subalgebras of $K[[t]]$. Using $q = e^t$, $p > 2$ and $r_t \geq \frac{2e}{p-1}$ one can show that

$$q \in \left(K[[t]]^\circ_{r_t}\right)^\times; \qquad \frac{1-q}{t} \in \left(K[[t]]^\circ_{r_t}\right)^\times; \qquad \frac{q-q^{-1}}{t} \in \left(K[[t]]^\circ_{r_t}\right)^\times.$$

To see this one shows that either of the expressions can we written as $a + f$ with $a \in \mathcal{O}^\times$ and $\nu(f) > 0$. Note that $p > 2$ is necessary since $\frac{q-q^{-1}}{t} = 2 + f$ with $\nu(f) > 0$ and thus for $p = 2$ we would have $\nu\left(\frac{q-q^{-1}}{t}\right) > 0$. We will later use that for $h \in \mathcal{O}$ with $\nu(h) \geq r_t$ there is a unique valuation increasing surjective K-algebra morphism

$$K[[t]]_{r_t} \longrightarrow K$$

sending t to h.

Definition 2.3.3. Let $r_t > \frac{2e}{p-1}$. Recall that for $\mu \in \mathbb{N}_0^4$ we defined

$$e^\mu = (a-1)^{\mu_1} b^{\mu_2} c^{\mu_3} (d-1)^{\mu_4}.$$

Similarly as in the case of $M_q(2,K)$ the set $\left\{e^\mu : \mu \in \mathbb{N}_0^4\right\}$ is $K[[t]]$-linearly independent in $M_q(2,t)$. For $2 \leq 2r \leq r_t$ we define a $K[[t]]_{r_t}$-subalgebra of $M_q(2,t)$ by setting

$$C_t^{an}(e,r,r_t) := \left\{ \sum_{\mu \in \mathbb{N}_0^4} a_\mu e^\mu \in M_q(2,t) : a_\mu \in K[[t]]_{r_t}, \quad \nu(a_\mu) + r|\mu| \to \infty \right\}.$$

On $C_t^{an}(e,r,r_t)$ we define a valuation ν_{e,r,r_t} by

$$\nu_{e,r,r_t}\left(\sum_\mu a_\mu e^\mu\right) = \min\{\nu(a_\mu) + r|\mu|\}.$$

2.3.4. Note that $C_t^{an}(e,r,r_t)$ is not complete. But as in the case of $M_q(2,K)$ one can show that ν_{e,r,r_t} is multiplicative for $2 \leq 2r < r_t$. If in addition $\nu(\log(q)) \geq r_t$ we have a valuation increasing surjection

$$C_t^{an}(e,r,r_t) \longrightarrow C_q^{an}(e,r)$$

which sends t to $\log q$.

Chapter 3

The quantum distribution algebra

In the previous chapter we discussed the Hopf algebra of quantum locally analytic functions $C_q^{la}(H,K)$. Now we want to describe its dual, the algebra of quantum locally analytic distributions.
We will show in Theorem 3.4.18 that $D_q(H,K) := C_q^{la}(H,K)'_b$ is a Fréchet Stein algebra. Now we describe the strategy of the proof. Since

$$C_q^{la}(H,K) = \varinjlim_{2r<\nu(1-q)} C_q^{la}(H,r)$$

is of compact type we know that

$$C_q^{la}(H,K)'_b = \varprojlim_{2r<\nu(1-q)} C_q^{la}(H,r)'_b.$$

Because of $C_q^{la}(H,r) = \bigoplus_i C_q^{la}(g_i,r)$ we can conclude that

$$C_q^{la}(H,r)'_b = \bigoplus_i C_q^{an}(g_i,r)'_b.$$

In section 3.4 we will show that there exist elements $\delta_{g_i} \in C_q^{an}(H,r)'_b$ such that $C_q^{an}(g_i,r)'_b = \delta_{g_i} C_q^{an}(e,r)'_b$ for e the identity matrix. This observation will enable us to reduce the problem to $C_q^{an}(e,r)'_b$. In order to analyze $C_q^{an}(e,r)'_b$ we will describe it as a completion of the quantum enveloping algebra $U_q(\mathfrak{gl}_2,K)$. This will simplify the problem since there is an explicit description of $U_q(\mathfrak{gl}_2,K)$ by generators and relations. To describe $C_q^{an}(e,r)'_b$ we will first propose a vector space candidate $D_q^{an}(e,r)$ for $C_q^{an}(e,r)'_b$ and show that the canonical bracket

$$U_q(\mathfrak{gl}_2,K) \times M_q(2,K) \longrightarrow K$$

extends to a bracket

$$D_q^{an}(e,r)\widehat{\otimes}C_q^{an}(e,r) \longrightarrow K.$$

Then we use some estimates about the bracket that we show in section 3.2.2 to conclude that $D_q^{an}(e,r) = C_q^{an}(e,r)'_b$. For technical reasons we will not directly analyze the bracket $D_q^{an}(e,r)\widehat{\otimes}C_q^{an}(e,r) \longrightarrow K$ but we will analyze a t-adic version of it.
Using the theory of partial divided powers we will define algebras

$$D_q^m(e,r) \subseteq D_q^{an}(e,r).$$

Also here we will first consider a t-adic counterpart in order to show that $D_q^m(e,r)$ is a K-Banach algebra and to show some estimates concerning commutator relations.
In **3.3.12** we will see that we can find a sequence $\{(r_n, m_n)\}_{n\in\mathbb{N}} \subset (\mathbb{Q},\mathbb{N})^{\mathbb{N}}$ such that $\{D_q^{an}(e,r) : 2r < \nu(1-q)\}$ and $\{D_q^{m_n}(e,r_n) : n \in \mathbb{N}\}$ are cofinal systems. We will see that $\{D_q^{m_n}(e,r_n) : n \in \mathbb{N}\}$ is a system consisting of Noetherian K-Banach algebras with right flat transition maps. This will be used in section 3.4 to show that $D_q(H,K)$ is a Fréchet Stein algebra.

3.1 q-calculus

Here we will collect some definitions and statements about q-calculus that will be helpful for this chapter. A more detailed treatment of q-calculus can be found in [KS97] chapter 2. Recall that $e = \nu(p)$ and that we always will assume that $q \notin \{1,-1\}$.

Definition 3.1.1. For $n \in \mathbb{N}_0$ and $q \in \mathcal{O}^\times\backslash\{\pm 1\}$ we define $[n] := \frac{q^n - q^{-n}}{q - q^{-1}}$ and $[n]! := \prod_{i=1}^n [i]$. Note that the empty product is equal to one and thus $[0]! = 1$.

Lemma 3.1.2. *Let $N \in \mathbb{N}$. If $\nu(1-q) > \frac{e}{p-1}$ then*

$$\nu([N]) = \nu(N).$$

In particular q is not a root of unity for $\nu(1-q) > \frac{e}{p-1}$.

Proof. Since $\nu(1-q) > \frac{e}{p-1}$ we know that $\nu(\log(q)) > \frac{e}{p-1}$, see e.g. the proof of [Gou97] Proposition 4.5.8. The same proposition implies then that $\nu\left(1 - e^{n\log(q)}\right) = \nu(n) + \nu(\log(q))$ for $n \in \mathbb{Z}$.

Hence $\nu(1-q^n) = \nu(n) + \nu(\log(q))$ and we can conclude that

$$
\begin{aligned}
\nu\left(\frac{q^N - q^{-N}}{q - q^{-1}}\right) &= \nu\left(q^{-N+1}\frac{1-q^{2N}}{1-q^2}\right) \\
&= \nu(2) + \nu(N) + \nu(\log(q)) - \nu(2) - \nu(\log(q)) \\
&= \nu(N).
\end{aligned}
$$

□

Definition 3.1.3. We define the symbol

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})$$

and $(a;q)_0 = 1$.

Lemma 3.1.4 ([KS97] section 2.1)**.** *It is true that* $[n]! = q^{\frac{-n(n-1)}{2}}\frac{(q^2;q^2)_n}{(1-q^2)^n}$ *and in particular*

$$\nu([n]!) = \nu\left(\frac{(q^2;q^2)_n}{(1-q^2)^n}\right).$$

Lemma 3.1.5. *Let* $x_1, ..., x_n$ *be elements in some ring* R*. Assume there exists* $q \in R\backslash\{1\}$ *such that* $x_j x_i = q x_i x_j$ *for* $i < j$*. Let* S_n *be the symmetric group for* n *elements. Then*

$$\sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)} = \frac{(q;q)_n}{(1-q)^n} x_1 \cdots x_n.$$

In particular $\frac{(q;q)_n}{(1-q)^n}$ *is a polynomial in* q*.*

Proof. We will use induction on n. For $n = 1$ the statement is obviously true. Recall that we have an injective group homomorphism $\iota : S_n \to S_{n+1}$ defined by $\iota\sigma(j) = \sigma(j)$ if $j \leq n$ and $\iota\sigma(n+1) = n+1$. Via ι we treat S_n as a subgroup of S_{n+1}. Let $t_i \in S_{n+1}$ for $i \in \{1, ..., n\}$ be the transposition interchanging i and $n+1$ and let $t_{n+1} = \mathrm{id}_{S_{n+1}}$. We then have $S_{n+1} = \coprod_{i=1}^{n+1} S_n t_i$. For $\sigma \in S_n$ we get

$$\sigma t_i(j) = \sigma(j) \text{ if } n+1 \neq j \neq i; \qquad \sigma t_i(i) = n+1; \qquad \sigma t_i(n+1) = \sigma(i).$$

Define $s_i \in S_n$ by

$$s_i(j) = j;\ \forall j < i; \qquad s_i(j) = j+1;\ \forall i \leq j < n; \qquad s_i(n) = i.$$

By using $x_{n+1} x_{\sigma t_i(j)} = q x_{\sigma t_i(j)} x_{n+1}$ for $j \neq i$ we obtain

$$x_{\sigma t_i(1)} \cdots x_{\sigma t_i(n+1)} = q^{n+1-i} x_{\sigma s_i(1)} \cdots x_{\sigma s_i(n)} x_{n+1}$$

since $\sigma t_i(i) = n+1$. Hence we can compute

$$\begin{aligned}\sum_{\sigma\in S_n} x_{\sigma t_i(1)}\cdots x_{\sigma t_i(n+1)} &= q^{n+1-i}\sum_{\sigma\in S_n} x_{\sigma s_i(1)}\cdots x_{\sigma s_i(n)}x_{n+1}\\ &= q^{n+1-i}\sum_{\sigma\in S_n} x_{\sigma(1)}\cdots x_{\sigma(n)}x_{n+1}\end{aligned}$$

and using $S_{n+1} = \coprod_{i=1}^{n+1} S_n t_i$ and induction on n we obtain

$$\begin{aligned}\sum_{\sigma\in S_{n+1}} x_{\sigma(1)}\cdots x_{\sigma(n+1)} &= \sum_{i=1}^{n+1}\sum_{\sigma\in S_n} x_{\sigma t_i(1)}\cdots x_{\sigma t_i(n+1)}\\ &= \sum_{i=1}^{n+1} q^{n+1-i}\frac{(q;q)_n}{(1-q)^n}x_1\cdots x_{n+1}\\ &= \frac{1-q^{n+1}}{1-q}\frac{(q;q)_n}{(1-q)^n}x_1\cdots x_{n+1}\\ &= \frac{(q;q)_{n+1}}{(1-q)^{n+1}}x_1\cdots x_{n+1}.\end{aligned}$$

□

Lemma 3.1.6. *Let $XY = qYX$. Then*

$$(X+Y)^n = \sum_{k=0}^{n}\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}Y^kX^{n-k}$$

Proof. This is [KS97] 2.1.2 Proposition 2. □

3.2 t-adic quantum enveloping algebras and some completions

3.2.1 The t-adic algebra U_t

In order to describe the quantum distribution algebra we will need certain completions of the quantum enveloping algebra. These completions can be seen as quotients of subalgebras of U_t. This is the reason why we investigate the t-adic algebra U_t.

First we will recall the definition of U_t as for example given in [KS97] sections 1.2.10 and 6.1.3.

Definition 3.2.1. Let $K[t]\{H_1, F, E, H_2\}$ be the noncommutative polynomial ring in the variables H_1, F, E, H_2 over $K[t]$. Let $K[[t]]\langle H_1, F, E, H_2\rangle$ be its t-adic completion. In this ring the sum

$$\sum_{n\geq 0}\frac{(tX)^n}{n!}$$

exists for $X \in K[[t]]\langle H_1, F, E, H_2\rangle$ and will be denoted by e^{tX}. We have the equality $e^t - e^{-t} = t(2 + tf)$ for some $f \in K[[t]]$ and hence $e^t - e^{-t} = tu$ with $u \in K[[t]]^\times$. Because $e^{-t(H_1-H_2)} - e^{t(H_1-H_2)}$ is divisible by t the expression

$$\frac{e^{-t(H_1-H_2)} - e^{t(H_1-H_2)}}{e^t - e^{-t}}$$

is well defined.
U_t, the quantum t-adic enveloping algebra for $\mathrm{GL}(2, K)$, is defined to be the quotient of $K[[t]]\langle H_1, F, E, H_2\rangle$ by the relations

$$[H_i, H_j] = 0, \qquad [H_i, F] = (-1)^{i+1} F,$$
$$[H_i, E] = (-1)^i E, \qquad [E, F] = \frac{e^{-t(H_1-H_2)} - e^{t(H_1-H_2)}}{e^t - e^{-t}}.$$

U_t is a complete t-adic K-algebra and as K vector space

$$U_t = \left\{ \sum_{(l,j)\in\mathbb{N}_0^4\times\mathbb{N}_0} a_{l,j} t^j H_1^{l_1} F^{l_2} E^{l_3} H^{l_4} : a_{l,j} \in K,\ \lim_{|l|\to\infty} min\{j : a_{l,j} \neq 0\} = \infty \right\}.$$

We have continuous $K[[t]]$-algebra morphisms

$$\Delta : U_t \to U_t \widehat{\otimes} U_t; \qquad \epsilon : U_t \to K[[t]]; \qquad S : U_t \to U_t^{op}$$

giving U_t the structure of a topological Hopf algebra. On the generators these are given by

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i; \qquad \Delta(E) = E \otimes e^{-t(H_1-H_2)} + 1 \otimes E;$$
$$\Delta(F) = F \otimes 1 + e^{t(H_1-H_2)} \otimes F; \qquad \epsilon(H_i) = \epsilon(E) = \epsilon(F) = 0;$$
$$S(E) = -E e^{t(H_1-H_2)}; \qquad S(F) = -e^{-t(H_1-H_2)} F;$$
$$S(H_i) = -H_i.$$

Let $K_i := e^{tH_i}$ for $i \in \{1, 2\}$. One easily deduces

$$\Delta(K_i) = K_i \otimes K_i; \qquad \epsilon(K_i) = 1; \qquad S(K_i) = K_i^{-1}.$$

Moreover

$$K_i E = q^{(-1)^i} E K_i; \qquad K_i F = q^{(-1)^{i+1}} F K_i.$$

where $q := e^t \in K[[t]]^\times$.

Definition 3.2.2. [[KS97] p.163.] Let L be a field and let $q \in L^\times - \{1, -1\}$. The quantum enveloping algebra of $\mathfrak{gl}(2, L)$ which we denote by $U_q(\mathfrak{gl}_2, L)$ is defined to be the unital associative L-algebra with generators

$$K_1, K_2, K_1^{-1}, K_2^{-1}, E, F$$

subject to the following relations.

$$\begin{aligned} &[K_1^{\pm}, K_2^{\pm}] = 0; && K_1K_1^{-1} = K_1^{-1}K_1 = 1; \\ &K_2K_2^{-1} = K_2^{-1}K_2 = 1; && K_iF = q^{(-1)^{i+1}}FK_i; \\ &K_iE = q^{(-1)^i}EK_i; && [E, F] = \frac{K_1^{-1}K_2 - K_1K_2^{-1}}{q - q^{-1}}. \end{aligned}$$

$U_q(\mathfrak{gl}_2, L)$ is a bialgebra with coalgebra structure given on generators by

$$\begin{aligned} &\Delta(K_i) = K_i \otimes K_i; && \Delta(E) = E \otimes K_1^{-1}K_2 + 1 \otimes E; \\ &\Delta(F) = F \otimes 1 + K_1K_2^{-1} \otimes F; && \epsilon(E) = \epsilon(F) = 0; \\ &\epsilon(K_i) = 1. && \end{aligned}$$

3.2.2 The bracket

In this section we will show some estimates concerning a bracket of topological algebras

$$U_t \widehat{\otimes} M_q(2, t) \longrightarrow K[[t]].$$

The only result of this section that we will need later on is Theorem 3.2.25 and everything before is just auxiliary computations.

Lemma 3.2.3. *There is a unique continuous $K[[t]]$-bilinear bracket*

$$\langle \cdot, \cdot \rangle : \ U_t \widehat{\otimes} M_q(2, t) \longrightarrow K[[t]]$$

with the property

$$\langle fg, h \rangle = \langle f \otimes g, \Delta(h) \rangle; \qquad \langle w, yz \rangle = \langle \Delta(w), y \otimes z \rangle; \tag{3.2.1}$$

$$\langle 1, h \rangle = \epsilon(h); \qquad \langle w, 1 \rangle = \epsilon(w). \tag{3.2.2}$$

for $f, g, w \in U_t$ and $h, y, z \in M_q(2, t)$ and which on generators is given by

$$\begin{aligned} &\langle H_1, x \rangle = \delta_{a,x}; && \langle H_2, x \rangle = \delta_{d,x}; \\ &\langle F, x \rangle = \delta_{b,x}; && \langle E, x \rangle = \delta_{c,x}. \end{aligned}$$

for $x \in \{a, b, c, d\}$. Moreover

$$\langle K_i, a \rangle = q^{\delta_{i,1}}; \qquad \langle K_i, d \rangle = q^{\delta_{i,2}};$$

$$\langle K_i, c \rangle = \langle K_i, b \rangle = 0; \tag{3.2.3}$$

Proof. Let $L = K((T))$ be the field of Laurent series with multiplicative valuation ν_L defined by

$$\nu_L\left(\sum_{l\in\mathbb{Z}} a_l t^l\right) := \min\{l : a_l \neq 0\} > -\infty.$$

Let $U_q(\mathfrak{gl}_2, L)$ for $q = e^t$ be the usual quantum enveloping algebra, see Definition 3.2.2. In [KS97] Chapter 9.4 Theorem 18 it is shown that there is a bracket of bialgebras $U_q(\mathfrak{gl}_2, L) \times M_q(2, L) \to L$ with

$$\begin{aligned} &\langle F, x\rangle = \delta_{b,x}; && \langle E, x\rangle = \delta_{c,x};\\ &\langle K_i, a\rangle = q^{\delta_{i,1}}; && \langle K_i, d\rangle = q^{\delta_{i,2}};\\ &\langle K_i, c\rangle = \langle K_i, b\rangle = 0; && \end{aligned}$$

for $x \in \{a, b, c, d\}$. Note that the set $\{K_1^l F^n E^m K_2^k : l, k \in \mathbb{Z} \text{ and } m, n \in \mathbb{N}_0\}$ is an L-basis of $U_q(\mathfrak{gl}_2, L)$ by [KS97] Section 3.1.1 Proposition 1. But then also

$$\left\{\left(\frac{K_1^\pm - 1}{t}\right)^l F^n E^m \left(\frac{K_2^\pm - 1}{t}\right)^k : l, k, m, n \in \mathbb{N}_0\right\}$$

is an L-basis of $U_q(\mathfrak{gl}_2, L)$. Since L carries the t-adic valuation ν_L we can endow $U_q(\mathfrak{gl}_2, L)$ with a valuation ν_U defined by

$$\nu_U\left(\sum a_{l,k,m,n}\left(\frac{K_1^\pm - 1}{t}\right)^l F^n E^m \left(\frac{K_2^\pm - 1}{t}\right)^k\right) := \min\left\{\nu_L\left(a_{l,k,m,n}\right)\right\}.$$

Claim 1: ν_U is submultiplicative. To show this, it is enough to show that for two monomials

$$\begin{aligned} m &:= \left(\frac{K_1^\pm - 1}{t}\right)^{l_1} F^{n_1} E^{m_1} \left(\frac{K_2^\pm - 1}{t}\right)^{k_1}\\ m' &= \left(\frac{K_1^\pm - 1}{t}\right)^{l_2} F^{n_2} E^{m_2} \left(\frac{K_2^\pm - 1}{t}\right)^{k_2} \end{aligned}$$

and

$$mm' = \sum a_{l,k,m,n}\left(\frac{K_1^\pm - 1}{t}\right)^l F^n E^m \left(\frac{K_2^\pm - 1}{t}\right)^k$$

we have that $\nu_L(a_{l,k,m,n}) \geq 0$. In a very similar fashion as in the proof of

Lemma 2.1.6 this follows from

$$
\begin{aligned}
&\frac{K_2^{\pm}-1}{t}\frac{K_1^{\pm}-1}{t} = \frac{K_1^{\pm}-1}{t}\frac{K_2^{\pm}-1}{t}\\
&F\frac{K_i^{\pm}-1}{t} = q^{(-1)^i}\frac{K_i^{\pm}-1}{t}F + \frac{q^{(-1)^i}-1}{t}F\\
&E\frac{K_i^{\pm}-1}{t} = q^{(-1)^{i+1}}\frac{K_i^{\pm}-1}{t}E + \frac{q^{(-1)^{i+1}}-1}{t}E\\
&[E,F] = \frac{t^2}{q-q^{-1}}\left(\frac{K_1^{-1}-1}{t}\frac{K_2-1}{t} - \frac{K_1-1}{t}\frac{K_2^{-1}-1}{t}\right)\\
&\quad \frac{t}{q-q^{-1}}\left(\frac{K_1^{-1}-1}{t} + \frac{K_2-1}{t} - \frac{K_1-1}{t} - \frac{K_2^{-1}-1}{t}\right)
\end{aligned}
$$

since $\nu_L\left(\frac{q^{(-1)^i}-1}{t}\right) \geq 0$, $\nu_L\left(\frac{t^2}{q-q^{-1}}\right) \geq 0$ and $\nu_L\left(\frac{t}{q-q^{-1}}\right) \geq 0$. The latter inequalities can easily be seen by using the definition $q = e^t$.

Claim 2: The map $\Delta : U_q(\mathfrak{gl}_2, L) \longrightarrow U_q(\mathfrak{gl}_2, L) \otimes U_q(\mathfrak{gl}_2, L)$ is valuation increasing.

Since ν_U is submultiplicative also the valuation on $U_q(\mathfrak{gl}_2, L) \otimes U_q(\mathfrak{gl}_2, L)$ is submultiplicative. Thus we only have to show that for $X \in \left\{\frac{K_i^{\pm}-1}{t}, E, F\right\}$ there exist $a_{j,k} \in L$ with $\nu_L(a_{j,k}) \geq 0$ and $x_{j,k} \in \left\{\frac{K_i^{\pm}-1}{t}, E, F\right\}$ such that

$$\Delta(X) = \sum_j a_{1,k}x_{1,k} \otimes a_{2,k}x_{2,k}.$$

But this follows from

$$
\begin{aligned}
\Delta(E) &= E \otimes t^2\frac{K_1^{-1}-1}{t}\frac{K_2-1}{t} + E \otimes t\frac{K_1^{-1}-1}{t}\\
&\quad + E \otimes t\frac{K_2-1}{t} + E \otimes 1 + 1 \otimes E\\
\Delta(F) &= F \otimes 1 + t^2\frac{K_1-1}{t}\frac{K_2^{-1}-1}{t} \otimes F + t\frac{K_1-1}{t} \otimes F\\
&\quad + t\frac{K_2^{-1}-1}{t} \otimes F + 1 \otimes F\\
\Delta\left(\frac{K_i^{\pm}-1}{t}\right) &= t\frac{K_i^{\pm}-1}{t} \otimes \frac{K_i^{\pm}-1}{t} + 1 \otimes \frac{K_i^{\pm}-1}{t}\\
&\quad + \frac{K_i^{\pm}-1}{t} \otimes 1.
\end{aligned}
$$

Let ν_M be the valuation on $M_q(2, L)$ defined by

$$\nu_M\left(\sum a_{i,k,l,m}a^ib^kc^ld^m\right) = \min\{\nu_L(a_{i,k,l,m})\}.$$

In a similar way as in the case of $U_q(\mathfrak{gl}_2, L)$ one can show that this valuation is submultiplicative and the comultiplication is valuation increasing. Here the calculations are much easier and thus we omit them. It can also easily be shown that the counit and the unit map of both algebras are valuation increasing.

Claim 3: The Bracket $\langle \cdot, \cdot \rangle : U_q(\mathfrak{gl}_2, L) \otimes M_q(2, L) \to L$ is valuation increasing. Since it is L-bilinear it is enough to show that for $\mu, \eta \in \mathbb{N}_0^4$ we have that

$$\left\langle \left(\frac{K_1^{\pm} - 1}{t}\right)^{\eta_1} F^{\eta_2} E^{\eta_3} \left(\frac{K_2^{\pm} - 1}{t}\right)^{\eta_4}, a^{\mu_1} b^{\mu_2} c^{\mu_3} d^{\mu_4} \right\rangle \in \mathcal{O}_L.$$

Because of Claim 1, Claim 2 and

$$\begin{aligned}
&\left\langle \left(\frac{K_1^{\pm} - 1}{t}\right)^{\eta_1} F^{\eta_2} E^{\eta_3} \left(\frac{K_2^{\pm} - 1}{t}\right)^{\eta_4}, a^{\mu_1} b^{\mu_2} c^{\mu_3} d^{\mu_4} \right\rangle \\
&= \left\langle \Delta^{|\mu|-1} \left(\left(\frac{K_1^{\pm} - 1}{t}\right)^{\eta_1} F^{\eta_2} E^{\eta_3} \left(\frac{K_2^{\pm} - 1}{t}\right)^{\eta_4} \right), a^{\otimes \mu_1} b^{\otimes \mu_2} c^{\otimes \mu_3} d^{\otimes \mu_4} \right\rangle
\end{aligned}$$

this can be reduced to

$$\left\langle \left(\frac{K_1^{\pm} - 1}{t}\right)^{\delta_1} F^{\delta_2} E^{\delta_3} \left(\frac{K_2^{\pm} - 1}{t}\right)^{\delta_4}, x \right\rangle \in \mathcal{O}_L$$

for $\delta \in \mathbb{N}_0^4$ and $x \in \{a, b, c, d\}$. Because of

$$\begin{aligned}
&\left\langle \left(\frac{K_1^{\pm} - 1}{t}\right)^{\delta_1} F^{\delta_2} E^{\delta_3} \left(\frac{K_2^{\pm} - 1}{t}\right)^{\delta_4}, x \right\rangle \\
&= \left\langle \left(\frac{K_1^{\pm} - 1}{t}\right)^{\otimes \delta_1} F^{\otimes \delta_2} E^{\otimes \delta_3} \left(\frac{K_2^{\pm} - 1}{t}\right)^{\otimes \delta_4}, \Delta^{|\delta|-1}(x) \right\rangle
\end{aligned}$$

we only need to show that

$$\langle y, x \rangle \in \mathcal{O}_L$$

for $y \in \left\{ \frac{K_i^{\pm}-1}{t}, E, F \right\}$ and $x \in \{a, b, c, d\}$. But this follows from

$$\begin{aligned}
\langle E, x \rangle &= \delta_{c,x}; & \left\langle \frac{K_1^{\pm} - 1}{t}, x \right\rangle &= \delta_{a,x} \frac{q^{\pm} - 1}{t}; \\
\langle F, x \rangle &= \delta_{b,x}; & \left\langle \frac{K_2^{\pm} - 1}{t}, x \right\rangle &= \delta_{d,x} \frac{q^{\pm} - 1}{t}.
\end{aligned}$$

The claims 1-3 imply that the bracket can be extended to a continuous bracket of L-Banach bialgebras on the completions

$$\widehat{U_q(\mathfrak{gl}_2, L)}^{\nu_U} \widehat{\otimes} \widehat{M_q(2, L)}^{\nu_M} \longrightarrow L.$$

Define

$$H_i := \frac{1}{t}\log(K_i) = \frac{1}{t}\log\left(1 + t\frac{K_i - 1}{t}\right)$$
$$= \frac{1}{t}\sum_{k\geq 1}(-1)^{k+1}\frac{t^k\left(\frac{K_i-1}{t}\right)^k}{k} \in \widehat{U_q(\mathfrak{gl}_2, L)}^{\nu_U}.$$

Let U be the $K[[t]]$ subalgebra of $\widehat{U_q(\mathfrak{gl}_2, L)}^{\nu_U}$ generated by H_1, H_2, E, F. Because $\nu_U(E) = \nu_U(F) = 0$ and $\nu(H_i) \geq 0$ the completion $\widehat{U}$ of U by ν_U is a t-adically complete $K[[t]]$-subalgebra of $\widehat{U_q(\mathfrak{gl}_2, L)}^{\nu_U}$. But since E, F, H_1, H_2 respect the relations in Definition 3.4 we can conclude that we have continuous $K[[t]]$-algebra morphism $\iota : U_t \to \widehat{U} \subset \widehat{U_q(\mathfrak{gl}_2, L)}^{\nu_U}$.
We obviously have a continuous inclusion $M_q(2,t) \subset \widehat{M_q(2, L)}^{\nu_M}$ and thus we have a continuous $K[[t]]$-linear bracket

$$\langle\cdot,\cdot\rangle : U_t\widehat{\otimes}M_q(2,t) \longrightarrow L.$$

Since $\Delta(\iota(X)) = (\iota\otimes\iota)(\Delta(X))$, $\epsilon(\iota(X)) = \iota(\epsilon(X))$ for $X \in \{H_1, H_2, E, F\}$ and likewise for $M_q(2,t)$ the relations (3.2.1) are satisfied. Using the same argument as in the proof of Claim 3 we can conclude that the image is in fact contained in $K[[t]] = \mathcal{O}_L$. $\square$

Definition 3.2.4. We denote the identity element in $\mathrm{GL}(2,K)$ by e, i.e.

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Recall that for $\mu = (\mu_a, \mu_b, \mu_c, \mu_d) \in \mathbb{N}_0^4$ we defined

$$e^\mu := (a-1)^{\mu_a} b^{\mu_b} c^{\mu_c} (d-1)^{\mu_d}.$$

3.2.5. For $SL_q(2,t)$ there is a formula for the bracket see e.g. [KS97] chapter 4 Proposition 22. But for $M_q(2,t)$ such a formula doesn't seem to be known. Also one usually considers the basis $\{a^{\mu_a}b^{\mu_b}c^{\mu_c}d^{\mu_d}\}$ whereas we have to use the basis $\{e^\mu : \mu \in \mathbb{N}_0^4\}$ since we would like to describe the dual of $C_q^{an}(e,r)$. This is why we can't just use results stated in the literature.

Lemma 3.2.6. *Let m be any (noncommutative) monomial in the variables $a-1, b, c, d-1$ and for $x \in \{a-1, b, c, d-1\}$ let μ_x be the number of times that x occurs in m. Then*

$$\langle H_1, m\rangle = \delta_{1,\mu_a}\delta_{0,\mu_b}\delta_{0,\mu_c}\delta_{0,\mu_d}$$
$$\langle H_2, m\rangle = \delta_{0,\mu_a}\delta_{0,\mu_b}\delta_{0,\mu_c}\delta_{1,\mu_d}.$$

Moreover, for a finite set $\{z_1, ..., z_n\} \subset U_t$

$$\left\langle K_i, \prod_{j=1}^{n} z_j \right\rangle = \prod_{j=1}^{n} \langle K_i, z_j \rangle$$

Proof. Note that $\langle 1, x \rangle = \epsilon(x) = 0$ for $x \in \{a-1, b, c, d-1\}$. The formulas follow from $\Delta^{n-1} H_i = \sum_{j=1}^{n} 1^{\otimes^{j-1}} \otimes H_i \otimes 1^{\otimes^{n-j}}$ resp. $\Delta^{n-1} K_i = K_i^{\otimes^n}$ and the property $\langle X, z \cdot w \rangle = \langle \Delta X, z \otimes w \rangle$. $\square$

Lemma 3.2.7. *For* $\mu = (\mu_a, \mu_b, \mu_c, \mu_d) \in \mathbb{N}_0^4$, $x \in \{a, b, c, d\}$ *and* $n \in \mathbb{Z}$ *with* $n + \mu_x \geq 0$ *we define*

$$e^{\mu}_{x,n} := (a-1)^{\mu_a + n\delta_{a,x}} b^{\mu_b + n\delta_{b,x}} c^{\mu_c + n\delta_{c,x}} (d-1)^{\mu_d + n\delta_{d,x}}.$$

For any $X \in U_t$ *the following formulas hold.*

$$\begin{aligned}
\langle H_1 X, e^{\mu} \rangle &= (\mu_a + \mu_b) \langle X, e^{\mu} \rangle + \mu_a \langle X, e^{\mu}_{a,-1} \rangle \\
\langle H_2 X, e^{\mu} \rangle &= (\mu_d + \mu_c) \langle X, e^{\mu} \rangle + \mu_d \langle X, e^{\mu}_{d,-1} \rangle \\
\langle X H_1, e^{\mu} \rangle &= (\mu_a + \mu_c) \langle X, e^{\mu} \rangle + \mu_a \langle X, e^{\mu}_{a,-1} \rangle \\
\langle X H_2, e^{\mu} \rangle &= (\mu_d + \mu_b) \langle X, e^{\mu} \rangle + \mu_d \langle X, e^{\mu}_{d,-1} \rangle.
\end{aligned}$$

Proof. We will only show the first equation since the other ones can be obtained analogously. We will use $\langle H_1 X, e^{\mu} \rangle = \langle H_1 \otimes X, \Delta(e^{\mu}) \rangle$. Recall that

$$\begin{aligned}
\Delta(a-1) =& (a-1) \otimes (a-1) + b \otimes c + 1 \otimes (a-1) + (a-1) \otimes 1 \\
\Delta(b) =& (a-1) \otimes b + b \otimes (d-1) + 1 \otimes b + b \otimes 1 \\
\Delta(c) =& c \otimes (a-1) + (d-1) \otimes c + 1 \otimes c + c \otimes 1 \\
\Delta(d-1) =& (d-1) \otimes (d-1) + c \otimes b + 1 \otimes (d-1) + (d-1) \otimes 1.
\end{aligned} \tag{3.2.4}$$

Let

$$\begin{aligned}
x_{a,1} &:= (a-1); & y_{a,1} &:= (a-1) & x_{a,2} &:= b; & y_{a,2} &= c; \\
x_{a,3} &:= 1; & y_{a,3} &:= (a-1); & x_{a,4} &:= (a-1); & y_{a,4} &= 1
\end{aligned}$$

i.e. $\Delta(a-1) = \sum_{i=1}^{4} x_{a,i} \otimes y_{a,i}$ by (3.2.4).
Let $x_{s,i}, y_{s,i}$ for $s \in \{b, c, (d-1)\}$ be defined similarly such that (3.2.4) reads as

$$\begin{aligned}
\Delta(a-1) &= \sum_{i=1}^{4} x_{a,i} \otimes y_{a,i}; & \Delta(b) &= \sum_{i=1}^{4} x_{b,i} \otimes y_{b,i}; \\
\Delta(c) &= \sum_{i=1}^{4} x_{c,i} \otimes y_{c,i}; & \Delta(d-1) &= \sum_{i=1}^{4} x_{d,i} \otimes y_{d,i}.
\end{aligned} \tag{3.2.5}$$

Since Δ is multiplicative the equations (3.2.5) imply that we can write $\Delta\left(e^{\mu}\right)$ as a sum with summands equal to

$$\left(\prod_{j=1}^{\mu_a} x_{a,i_j} \prod_{k=1}^{\mu_b} x_{b,i_k} \prod_{l=1}^{\mu_c} x_{c,i_l} \prod_{m=1}^{\mu_d} x_{d,i_m}\right) \otimes \left(\prod_{j=1}^{\mu_a} y_{a,i_j} \prod_{k=1}^{\mu_b} y_{b,i_k} \prod_{l=1}^{\mu_c} y_{c,i_l} \prod_{m=1}^{\mu_d} y_{d,i_m}\right)$$

where $i_j, i_k, i_l, i_m \in \{1, ..., 4\}$. By Lemma 3.2.6 we know that

$$\left\langle H_1, \prod_{j=1}^{\mu_a} x_{a,i_j} \prod_{k=1}^{\mu_b} x_{b,i_k} \prod_{l=1}^{\mu_c} x_{c,i_l} \prod_{m=1}^{\mu_d} x_{d,i_m} \right\rangle \neq 0$$

iff

$$\prod_{j=1}^{\mu_a} x_{a,i_j} \prod_{k=1}^{\mu_b} x_{b,i_k} \prod_{l=1}^{\mu_c} x_{c,i_l} \prod_{m=1}^{\mu_d} x_{d,i_m} = (a-1).$$

By (3.2.4) the only possibilities for the latter are

$$\begin{aligned}
&\left(\prod_{j=1}^{r-1} x_{a,3}\right) \cdot x_{a,1} \left(\prod_{j=r+1}^{\mu_a} x_{a,3}\right) \prod_{k=1}^{\mu_b} x_{b,3} \prod_{l=1}^{\mu_c} x_{c,3} \prod_{m=1}^{\mu_d} x_{d,3} \\
&\qquad = \left(\prod_{j=1}^{r-1} 1\right) \cdot (a-1) \left(\prod_{j=r+1}^{\mu_a} 1\right) \prod_{k=1}^{\mu_b} 1 \prod_{l=1}^{\mu_c} 1 \prod_{m=1}^{\mu_d} 1
\end{aligned}$$

for $1 \leq r \leq \mu_a$ and

$$\begin{aligned}
&\left(\prod_{j=1}^{s-1} x_{a,3}\right) \cdot x_{a,4} \left(\prod_{j=s+1}^{\mu_a} x_{a,3}\right) \prod_{k=1}^{\mu_b} x_{b,3} \prod_{l=1}^{\mu_c} x_{c,3} \prod_{m=1}^{\mu_d} x_{d,3} \\
&\qquad = \left(\prod_{j=1}^{s-1} 1\right) \cdot (a-1) \left(\prod_{j=s+1}^{\mu_a} 1\right) \prod_{k=1}^{\mu_b} 1 \prod_{l=1}^{\mu_c} 1 \prod_{m=1}^{\mu_d} 1
\end{aligned}$$

for $1 \leq s \leq \mu_a$ and

$$\begin{aligned}
&\prod_{j=1}^{\mu_a} x_{a,3} \left(\prod_{k=1}^{t-1} x_{b,3}\right) x_{b,1} \left(\prod_{k=t+1}^{\mu_b} x_{b,3}\right) \prod_{l=1}^{\mu_c} x_{c,3} \prod_{m=1}^{\mu_d} x_{d,3} \\
&\qquad = \prod_{j=1}^{\mu_a} 1 \left(\prod_{k=1}^{t-1} 1\right) (a-1) \left(\prod_{k=t+1}^{\mu_b} 1\right) \prod_{l=1}^{\mu_c} 1 \prod_{m=1}^{\mu_d} 1
\end{aligned}$$

for $1 \leq t \leq \mu_b$. In all three cases $\langle H_1, * \rangle = 1$ by Lemma 3.2.6. Hence we can conclude that

$$
\begin{aligned}
&\langle H_1 \otimes X, \Delta(e^\mu)\rangle \\
&= \sum_{r=1}^{\mu_a} \left\langle X, \left(\prod_{j=1}^{r-1} y_{a,3}\right) \cdot y_{a,1} \left(\prod_{j=r+1}^{\mu_a} y_{a,3}\right) \prod_{k=1}^{\mu_b} y_{b,3} \prod_{l=1}^{\mu_c} y_{c,3} \prod_{m=1}^{\mu_d} y_{d,3} \right\rangle \\
&+ \sum_{s=1}^{\mu_a} \left\langle X, \left(\prod_{j=1}^{s-1} y_{a,3}\right) \cdot y_{a,4} \left(\prod_{j=s+1}^{\mu_a} y_{a,3}\right) \prod_{k=1}^{\mu_b} y_{b,3} \prod_{l=1}^{\mu_c} y_{c,3} \prod_{m=1}^{\mu_d} y_{d,3} \right\rangle \\
&+ \sum_{t=1}^{\mu_b} \left\langle X, \prod_{j=1}^{\mu_a} y_{a,3} \left(\prod_{k=1}^{t-1} y_{b,3}\right) y_{b,1} \left(\prod_{k=t+1}^{\mu_b} y_{b,3}\right) \prod_{l=1}^{\mu_c} y_{c,3} \prod_{m=1}^{\mu_d} y_{d,3} \right\rangle \\
&= \sum_{r=1}^{\mu_a} \langle X, (a-1)^{\mu_a} b^{\mu_b} c^{\mu_c} (d-1)^{\mu_d}\rangle \\
&+ \sum_{s=1}^{\mu_a} \left\langle X, (a-1)^{\mu_a - 1} b^{\mu_b} c^{\mu_c} (d-1)^{\mu_d}\right\rangle \\
&+ \sum_{t=1}^{\mu_b} \langle X, (a-1)^{\mu_a} b^{\mu_b} c^{\mu_c} (d-1)^{\mu_d}\rangle \\
&= (\mu_a + \mu_b)\langle X, e^\mu\rangle + \mu_a \langle X, e^\mu_{a,-1}\rangle
\end{aligned}
$$

and we proved the first equation. □

Lemma 3.2.8. *We have*

$$
\begin{aligned}
&\langle (K_1 K_2^{-1})^l, b\rangle = 0 \\
&\langle (K_1 K_2^{-1})^l, c\rangle = 0 \\
&\langle (K_1 K_2^{-1})^l, a-1\rangle = q^l - 1 \\
&\langle (K_1^{-1} K_2)^l, a-1\rangle = q^{-l} - 1 \\
&\langle (K_1 K_2^{-1})^l, d-1\rangle = q^{-l} - 1 \\
&\langle (K_1^{-1} K_2)^l, d-1\rangle = q^l - 1.
\end{aligned}
$$

Proof. This is well known and can for example be found in [KS97] 4.4.1. □

Lemma 3.2.9. *We have*

$$
\begin{aligned}
&\langle (K_1 K_2^{-1})^l F^n E^m, (d-1)\rangle = \delta_{n,0}, \delta_{m,0}(q^{-l} - 1) \\
&\langle F^n E^m, b\rangle = \delta_{n,1}\delta_{m,0} \\
&\langle F^n E^m, c\rangle = \delta_{n,0}\delta_{m,1}.
\end{aligned}
$$

Proof. This is can for example be found in [KS97] 4.4.1. □

Proposition 3.2.10. *We have*

$$\langle F^n E^m, b^{n_b} c^{n_c} \rangle = \delta_{n,n_b} \delta_{m,n_c} q^{\frac{(n_b-1)n_b+(n_c-1)n_c}{2}} \frac{(q^2,q^2)_n (q^2,q^2)_m}{(1-q^2)^{n+m}}.$$

Proof. See e.g. [KS97] 4.4.1. □

Lemma 3.2.11. *For $X \in U_t$ we obtain the formulas*

$$\begin{aligned}
\langle K_1^{\pm} X, b^{\mu_b} c^{\mu_c} \rangle &= q^{\pm \mu_b} \langle X, b^{\mu_b} c^{\mu_c} \rangle \\
\langle K_2^{\pm} X, b^{\mu_b} c^{\mu_c} \rangle &= q^{\pm \mu_c} \langle X, b^{\mu_b} c^{\mu_c} \rangle \\
\langle X K_1^{\pm}, b^{\mu_b} c^{\mu_c} \rangle &= q^{\pm \mu_c} \langle X, b^{\mu_b} c^{\mu_c} \rangle \\
\langle X K_2^{\pm}, b^{\mu_b} c^{\mu_c} \rangle &= q^{\pm \mu_b} \langle X, b^{\mu_b} c^{\mu_c} \rangle.
\end{aligned}$$

Proof. We will only show the first equation and we will use

$$\langle K_1 X, b^{\mu_b} c^{\mu_c} \rangle = \langle K_1 \otimes X, \Delta (b^{\mu_b} c^{\mu_c}) \rangle.$$

Let

$$\begin{aligned}
x_{b,1} &= a; & y_{b,1} &= b; & x_{b,2} &= b; & y_{b,2} &= d \\
x_{c,1} &= c; & y_{c,1} &= a; & x_{c,2} &= d; & y_{c,2} &= c.
\end{aligned}$$

Then $\Delta(b) = \sum_{i=1}^2 x_{b,i} \otimes y_{b,i}$ and $\Delta(c) = \sum_{i=1}^2 x_{c,i} \otimes y_{c,i}$. Hence

$$\Delta (b^{n_b} c^{n_c}) = \sum_{i_j, i_k} \prod_{j=1}^{\mu_b} x_{b,i_j} \prod_{k=1}^{\mu_c} x_{c,i_k} \otimes \prod_{j=1}^{\mu_b} y_{b,i_j} \prod_{k=1}^{\mu_c} y_{c,i_k}.$$

where $i_j, i_k \in \{1,2\}$. By Lemma 3.2.6 we know that

$$\left\langle K_1, \prod_{j=1}^{\mu_b} x_{b,i_j} \prod_{k=1}^{\mu_c} x_{c,i_k} \right\rangle = \prod_{j=1}^{\mu_b} \langle K_1, x_{b,i_j} \rangle \cdot \prod_{k=1}^{\mu_c} \langle K_1, x_{c,i_k} \rangle.$$

Hence (3.2.3) implies that

$$\left\langle K_1, \prod_{j=1}^{\mu_b} x_{b,i_j} \prod_{k=1}^{\mu_c} x_{c,i_k} \right\rangle \neq 0$$

iff $x_{b,i_j} = a$ for all j and $x_{c,i_k} = d$ for all k i.e. $i_j = 1$ for all j and $i_k = 2$ for all k. In this case (3.2.3) implies that

$$\left\langle K_1, \prod_{j=1}^{\mu_b} x_{b,1} \prod_{k=1}^{\mu_c} x_{c,2} \right\rangle = \prod_{j=1}^{\mu_b} \langle K_1, a \rangle \cdot \prod_{k=1}^{\mu_c} \langle K_1, d \rangle = \prod_{j=1}^{\mu_b} q \prod_{k=1}^{\mu_c} 1 = q^{\mu_b}.$$

Using $\langle K_1 X, b^{\mu_b} c^{\mu_c} \rangle = \langle K_1 \otimes X, \Delta (b^{\mu_b} c^{\mu_c}) \rangle$ and the description of $\Delta (b^{\mu_b} c^{\mu_c})$

we thus can conclude that

$$\begin{aligned}\langle K_1 X, b^{\mu_b} c^{\mu_c}\rangle &= \left\langle K_1, \prod_{j=1}^{\mu_b} x_{b,1} \prod_{k=1}^{\mu_c} x_{c,2} \right\rangle \cdot \left\langle X, \prod_{j=1}^{\mu_b} y_{b,1} \prod_{k=1}^{\mu_c} y_{c,2} \right\rangle \\ &= q^{\mu_b} \left\langle X, b^{\mu_b} c^{\mu_c} \right\rangle .\end{aligned}$$

□

3.2.12. For $n \in \mathbb{N}_0$ we define

$$e_{i,n} := 1^{\otimes^{n-i-1}} \otimes E \otimes (K_1^{-1} K_2)^{\otimes^i}; \quad f_{i,n} := (K_1 K_2^{-1})^{\otimes^{n-i-1}} \otimes F \otimes 1^{\otimes^i}.$$

Then

$$\Delta^{n-1}(E) = \sum_{i=0}^{n-1} e_{i,n}; \qquad \Delta^{n-1}(F) = \sum_{i=0}^{n-1} f_{i,n}$$

and for $i < j$ it is true that $e_{j,n} e_{i,n} = q^2 e_{i,n} e_{j,n}$ and $f_{j,n} f_{i,n} = q^{-2} f_{i,n} f_{j,n}$.

Lemma 3.2.13. *We have*

$$\sum_{\sigma \in S_n} f_{\sigma(0),n} \cdots f_{\sigma(n-1),n} = \frac{(q^{-2}, q^{-2})_n}{(1-q^{-2})^n} f_{0,n} \cdots f_{n-1,n}$$
$$\sum_{\sigma \in S_n} e_{\sigma(0),n} \cdots e_{\sigma(n-1),n} = \frac{(q^2, q^2)_n}{(1-q^2)^n} e_{0,n} \cdots e_{n-1,n}.$$

Proof. Follows from Lemma 3.1.5. □

Lemma 3.2.14. *We have*

$$\langle f_{0,n} \cdots f_{n-1,n}, b^{\otimes^n} \rangle = q^{\frac{(n-1)n}{2}}$$
$$\langle e_{0,n} \cdots e_{n-1,n}, c^{\otimes^n} \rangle = q^{\frac{(n-1)n}{2}}.$$

Proof. With the help of Lemma 3.2.11 we compute

$$\begin{aligned}\langle f_{0,n} \cdots f_{n-1,n}, b^{\otimes^n} \rangle &= \langle (K_1 K_2^{-1})^{n-1} F, b \rangle \cdots \langle F, b \rangle \\ &= q^{n-1} \cdots q^0 \\ &= q^{\frac{(n-1)n}{2}}.\end{aligned}$$

□

Lemma 3.2.15. *We have*

$$\langle F^n E^m, a - 1 \rangle = \delta_{n,1} \delta_{m,1}$$

and for $m+n>0$ and $l \in \mathbb{Z}$ we obtain

$$\langle (K_1K_2^{-1})^l F^n E^m, a-1\rangle = q^l \delta_{n,1}\delta_{m,1}.$$

Proof. This is a consequence of [KS97] 4.4.1. $\square$

3.2.16. Recall that in Definition 2.3.2 we defined for $r_t > \frac{2e}{p-1}$

$$K[[t]]_{r_t} := \left\{ \sum_{i\geq 0} a_i t^i : a_i \in K \text{ and } \lim_{i\to\infty} (\nu(a_i) + ir_t) = \infty \right\} \subseteq K[[t]]$$

and on $K[[t]]_{r_t}$ we defined a multiplicative valuation ν by setting

$$\nu\left(\sum_i a_i t^i\right) = \min\{\nu(a_i) + ir_t\}.$$

Moreover we defined $K[[t]]_{r_t}^\circ := \{f \in K[[t]]_{r_t} : \nu(f) \geq 0\}$ and mentioned

$$q \in \left(K[[t]]_{r_t}^\circ\right)^\times; \qquad \frac{1-q}{t} \in \left(K[[t]]_{r_t}^\circ\right)^\times; \qquad \frac{t}{q-q^{-1}} \in \left(K[[t]]_{r_t}^\circ\right)^\times.$$

Lemma 3.2.17. *Fix $n \in \mathbb{N}$. Let $f_i := f_{i,n}$, $e_i := e_{i,n}$ and let $k \leq n$ and*

$$0 \leq i_1 < \cdots < i_k \leq n-1.$$

Moreover let $r \in \mathbb{Z}$ and set $i_0 = -1$ and $i_{k+1} = n$. Then

$$\left\langle \left((K_1K_2^{-1})^r\right)^{\otimes^n} f_{i_1}\cdots f_{i_k} e_{i_1}\cdots e_{i_k}, (a-1)^{\otimes^n} \right\rangle$$

can be written as

$$\prod_{l=1}^{k+1} (q^{-k+2l-2+r} - 1)^{i_l - i_{l-1} - 1} \prod_{l=1}^{k} q^{-k+2l-1+r}.$$

Consequently

$$\left\langle \left((K_1K_2^{-1})^r\right)^{\otimes^n} f_{i_1}\cdots f_{i_k} e_{i_1}\cdots e_{i_k}, (a-1)^{\otimes^n} \right\rangle \in t^{n-k} K[[t]]_{r_t}^\circ.$$

Proof. Let $R = K_1K_2^{-1}$. Since $f_i \in U_t^{\otimes^n}$ and $e_i \in U_t^{\otimes^n}$ and since the multiplication on $U_t^{\otimes^n}$ is component wise we can conclude that $f_{i_1}\cdots f_{i_k}$ resp.

$e_{i_1} \cdots e_{i_k}$ can be written as

$$\left(R^k\right)^{\otimes i_{k+1}-i_k-1} \otimes R^{k-1}F \quad \otimes \left(R^{k-1}\right)^{\otimes i_k - i_{k-1}-1} \quad \otimes R^{k-2}F \quad \otimes \cdots$$

$$\cdots \otimes RF \quad \otimes R^{\otimes i_2 - i_1 - 1} \quad \otimes F \quad \otimes 1^{\otimes i_1}$$

resp.

$$1^{\otimes i_{k+1}-i_k-1} \otimes E \quad \otimes \left(R^{-1}\right)^{\otimes i_k-i_{k-1}-1} \quad \otimes ER^{-1} \quad \otimes \cdots$$

$$\cdots \otimes ER^{2-k} \quad \otimes \left(R^{1-k}\right)^{\otimes i_2-i_1-1} \quad \otimes ER^{1-k} \quad \otimes \left(R^{-k}\right)^{\otimes i_1}.$$

Because of $RFE = FER$ the product $f_{i_1} \cdots f_{i_k} \cdot e_{i_1} \cdots e_{i_k}$ is equal to

$$\left(R^k\right)^{\otimes i_{k+1}-i_k-1} \otimes R^{k-1}FE \quad \otimes \left(R^{k-2}\right)^{i_k-i_{k-1}-1} \quad \otimes R^{k-3}FE \quad \cdots \tag{3.2.6}$$

$$\cdots \otimes R^{3-k}FE \quad \otimes \left(R^{2-k}\right)^{\otimes i_2-i_1-1} \quad \otimes R^{1-k}FE \quad \otimes \left(R^{-k}\right)^{\otimes i_1}.$$

Since we defined $i_{k+1} = n$ and $i_0 = -1$ we can compute with the help of Lemma 3.2.15 and (3.2.6)

$$\left\langle (R^r)^{\otimes n} f_{i_1} \cdots f_{i_k} e_{i_1} \cdots e_{i_k}, (a-1)^{\otimes n} \right\rangle$$

as

$$\begin{aligned}
&\prod_{l=1}^{k+1} \left\langle R^{-k+2l-2+r}, (a-1) \right\rangle^{i_l - i_{l-1}-1} \prod_{l=1}^{k} \left\langle R^{-k+2l-1+r} FE, a-1 \right\rangle \\
&= \prod_{l=1}^{k+1} \left(q^{-k+2l-2+r} - 1 \right)^{i_l - i_{l-1}-1} \prod_{l=1}^{k} q^{-k+2l-1+r}
\end{aligned}$$

Note that $q - 1 = tg$ for some $g \in K[[t]]_p^\circ$ and that $q \in K[[t]]_p^\circ$. Combining with $q^l - 1 = (q-1)\sum_{k=0}^{l-1} q^k$ for $l \in \mathbb{N}_0$ and with $\sum_{l=1}^{k+1}(i_l - i_{l-1} - 1) = n - k$ we can conclude the second claim. □

Lemma 3.2.18. *Let $L, m, n \in \mathbb{N}_0$ and $m + n > 0$. If $n \neq m$ or $n > L$ then*

$$\left\langle (K_1 K_2^{-1})^r F^n E^m, (a-1)^L \right\rangle = 0.$$

Let $L, n \in \mathbb{N}_0$ be such that $L \geq n$. Then

$$\left\langle (K_1 K_2^{-1})^r F^n E^n, (a-1)^L \right\rangle \in t^{L-n} \frac{(q^2, q^2)_n^2}{(1-q^2)^{2n}} K[[t]]_{r_t}^\circ.$$

Proof. Let $f_i := f_{i,L}$ and $e_i := e_{i,L}$. Then

$$
\begin{aligned}
&\left\langle (K_1K_2^{-1})^r F^n E^m, (a-1)^L \right\rangle \\
&= \left\langle \Delta^{L-1}\left((K_1K_2^{-1})^r F^n E^m\right), (a-1)^{\otimes^L} \right\rangle \\
&= \left\langle \left((K_1K_2^{-1})^r\right)^{\otimes^L} \left(\Delta^{L-1}(F)\right)^n \left(\Delta^{L-1}(E)\right)^m, (a-1)^{\otimes^L} \right\rangle \\
&= \left\langle \left((K_1K_2^{-1})^r\right)^{\otimes^L} \left(\sum_{i=0}^{L-1} f_i\right)^n \left(\sum_{i=0}^{L-1} e_i\right)^m, (a-1)^{\otimes^L} \right\rangle. \qquad (3.2.7)
\end{aligned}
$$

Because the bracket is compatible with the tensor product we know by **3.2.12** and equation (3.2.7) that $\left\langle (K_1K_2^{-1})^r F^n E^m, (a-1)^L \right\rangle$ is a sum of products with factors of the shape

$$\langle q^k (K_1K_2^{-1})^l F^f E^e, a-1\rangle$$

for some $k, e, f \in N_0$, $l \in \mathbb{Z}$. By Lemma 3.2.15 such a factor is zero if $f+e>0$ and $(f,e) \neq (1,1)$. Using **3.2.12** again we conclude that if $n \neq m$ or if $n > L$ then at least one factor of every product in $\left\langle (K_1K_2^{-1})^r F^n E^m, (a-1)^L \right\rangle$ satisfies $f+e>0$ and $(f,e) \neq (1,1)$ and thus

$$\left\langle (K_1K_2^{-1})^r F^n E^m, (a-1)^L \right\rangle = 0$$

in this case.
We are left to show the statement for $n = m$ and $n \leq L$. Note that by Lemma 3.1.5 we obtain for $0 \leq i_1 < \cdots < i_n \leq L-1$ the equations

$$
\begin{aligned}
\sum_{\sigma \in S_n} f_{i_{\sigma(1)}} \cdots f_{i_{\sigma(n)}} &= q^{-n(n-1)} \frac{(q^2,q^2)_n}{(1-q^2)^n} f_{i_1} \cdots f_{i_n} \\
\sum_{\sigma \in S_n} e_{i_{\sigma(1)}} \cdots e_{i_{\sigma(n)}} &= \frac{(q^2,q^2)_n}{(1-q^2)^n} e_{i_1} \cdots e_{i_n}
\end{aligned}
$$

Moreover because $\langle (K_1K_2^{-1})^l F^f E^e, a-1\rangle = 0$ if $f+e>0$ and $(f,e) \neq (1,1)$ we can conclude that

$$\langle f_{j_1} \cdots f_{j_n} e_{k_1} \cdots e_{k_n}, (a-1)^{\otimes^L}\rangle = 0$$

if $\{j_1, ..., j_n\} \neq \{k_1, ..., k_n\}$ or if there exist $w \neq z$ such that $j_w = j_z$. Let

$R = K_1 K_2^{-1}$. Thus using equation (3.2.7) we see that

$$\begin{aligned}
&\left\langle R^r F^n E^n, (a-1)^L \right\rangle \\
&= \left\langle (R^r)^{\otimes L} \left(\sum_{i=0}^{L-1} f_i \right)^n \left(\sum_{i=0}^{L-1} e_i \right)^n, (a-1)^{\otimes L} \right\rangle \\
&= \sum_{0 \le i_1 < \cdots < i_n < L} \sum_{\sigma \in S_n} \sum_{\tau \in S_n} \left\langle (R^r)^{\otimes L} f_{i_{\sigma(1)}} \cdots f_{i_{\sigma(n)}} e_{i_{\tau(1)}} \cdots e_{i_{\tau(n)}}, (a-1)^{\otimes L} \right\rangle \\
&= \sum_{0 \le i_1 < \cdots < i_n < L} q^{-n(n-1)} \frac{(q^2, q^2)_n^2}{(1-q^2)^{2n}} \left\langle (R^r)^{\otimes L} f_{i_1} \cdots f_{i_n} e_{i_1} \cdots e_{i_n}, (a-1)^{\otimes L} \right\rangle .
\end{aligned}$$

Therefore by Lemma 3.2.17

$$\left\langle (K_1 K_2^{-1})^r F^n E^n, (a-1)^L \right\rangle \in t^{L-n} \frac{(q^2, q^2)_n^2}{(1-q^2)^{2n}} K[[t]]_{r_t}^{\circ}.$$

□

Lemma 3.2.19. *For $n, m, L \in \mathbb{N}_0$ we have*

$$\langle F^n E^m, X(d-1)^L \rangle = \langle F^n E^m, X \rangle (q^m - 1)^L$$

and therefore

$$\left\langle F^n E^m, X(d-1)^L \right\rangle \in t^L \langle F^n E^m, X \rangle K[[t]]_{r_t}^{\circ}.$$

Proof. Let $n, m \in \mathbb{N}_0$, $m + n > 0$ and let $s \in \mathbb{Z}$. Lemma 3.2.9 tells us that then $\left\langle (K_1^{-1} K_2)^s F^n E^m, (d-1) \right\rangle = 0$ and $\langle (K_1^{-1} K_2)^s, d-1 \rangle = q^s - 1$. Thus

$$\begin{aligned}
\langle F^n E^m, X(d-1)^L \rangle &= \left\langle \left(\sum_{i=0}^{L} f_{i,L} \right)^n \left(\sum_{i=0}^{L} e_{i,L} \right)^m, X \otimes (d-1)^{\otimes L} \right\rangle \\
&= \langle F^n E^m \otimes [(K_1^{-1} K_2)^m]^{\otimes L}, X \otimes (d-1)^{\otimes L} \rangle \\
&= \langle F^n E^m, X \rangle (q^m - 1)^L.
\end{aligned}$$

□

Proposition 3.2.20. *Let $n, m \in \mathbb{N}_0$ and let $\mu = (\mu_a, \mu_b, \mu_c, \mu_d) \in \mathbb{N}_0^4$. Then*

$$\langle F^n E^m, e^{\mu} \rangle \in \delta_{n-\mu_b, m-\mu_c} t^{\mu_a + \mu_d + \frac{1}{2}(\mu_b - n + \mu_c - m)} \frac{(q^2, q^2)_n (q^2, q^2)_m}{(1-q^2)^{n+m}} K[[t]]_{r_t}^{\circ}$$

and

$$\langle F^n E^m, e^{\mu} \rangle = 0$$

if $\mu_a < n - \mu_b$ or $n - \mu_b < 0$.

Proof. We know from Lemma 3.2.19 that

$$\langle F^n E^m, e^\mu \rangle = (q^m - 1)^{\mu_d} \langle F^n E^m, (a-1)^{\mu_a} b^{\mu_b} c^{\mu_c} \rangle.$$

From Lemma 3.1.6 we know that

$$\begin{aligned}
(\Delta(F))^n &= (F \otimes 1 + K_1 K_2^{-1} \otimes F)^n \\
&= \sum_{k=0}^{n} \frac{(q^2, q^2)_n}{(q^2, q^2)_k (q^2, q^2)_{n-k}} (K_1 K_2^{-1})^k F^{n-k} \otimes F^k \\
(\Delta(E))^m &= (1 \otimes E + E \otimes K_1^{-1} K_2)^m \\
&= \sum_{k=0}^{m} \frac{(q^2, q^2)_m}{(q^2, q^2)_k (q^2, q^2)_{m-k}} E^{m-k} \otimes E^k (K_1^{-1} K_2)^{m-k} \qquad (3.2.8)
\end{aligned}$$

By Lemma 3.2.11 and Proposition 3.2.10 we see that

$$\left\langle \left(K_1 K_2^{-1}\right)^s F^k E^l, b^i c^j \right\rangle = \delta_{k,i} \delta_{l,j} q^o \frac{(q^2, q^2)_k (q^2, q^2)_l}{(1 - q^2)^{k+l}}$$

for some $o \in \mathbb{Z}$. Thus using also the equations (3.2.8) we see that the expression

$$\langle F^n E^m, (a-1)^{\mu_a} b^{\mu_b} c^{\mu_c} \rangle = \langle (\Delta(F^n) \Delta(E^n), (a-1)^{\mu_a} \otimes b^{\mu_b} c^{\mu_c} \rangle$$

vanishes if $n < \mu_b$ or $m < \mu_c$. Let

$$\gamma := \frac{(q^2, q^2)_n (q^2, q^2)_m}{(q^2, q^2)_{\mu_b} (q^2, q^2)_{n-\mu_b} (q^2, q^2)_{\mu_c} (q^2, q^2)_{m-\mu_c}}.$$

If $n \geq \mu_b$ and $m \geq \mu_c$ only the term

$$\gamma \left\langle (K_1 K_2^{-1})^{\mu_b} F^{n-\mu_b} E^{m-\mu_c}, (a-1)^{\mu_a} \right\rangle \cdot \left\langle (K_1^{-1} K_2)^{m-\mu_c} F^{\mu_b} E^{\mu_c}, b^{\mu_b} c^{\mu_c} \right\rangle$$

may be nonzero and it is zero if $n - \mu_b \neq m - \mu_c$ or if $n - \mu_b > \mu_a$ by Lemma 3.2.18. If it is nonzero then by Lemma 3.2.18 there exists $f \in K[[t]]_p^\circ$ such that

$$\left\langle (K_1 K_2^{-1})^{\mu_b} F^{n-\mu_b} E^{m-\mu_c}, (a-1)^{\mu_a} \right\rangle = t^{\mu_a + \mu_b - n} f \frac{(q^2, q^2)_{n-\mu_b}^2}{(1 - q^2)^{2(n-\mu_b)}}.$$

Lemma 3.2.10 implies

$$\left\langle (K_1^{-1} K_2)^{m-\mu_c} F^{\mu_b} E^{\mu_c}, b^{\mu_b} c^{\mu_c} \right\rangle = q^s \frac{(q^2, q^2)_{\mu_b} (q^2, q^2)_{\mu_c}}{(1 - q^2)^{\mu_b + \mu_c}}$$

for some $s \in \mathbb{Z}$. Thus it is enough to check that

$$g := \gamma q^s \frac{(q^2, q^2)_{\mu_b} (q^2, q^2)_{\mu_c}}{(1 - q^2)^{\mu_b + \mu_c}} \frac{(q^2, q^2)_{n-\mu_b}^2}{(1 - q^2)^{2(n-\mu_b)}} \frac{(1 - q^2)^{n+m}}{(q^2, q^2)_n (q^2, q^2)_m} \in K[[t]]_{r_t}^\circ.$$

But since $n - \mu_b = m - \mu_c$ we know that $g = q^s$. $\square$

Definition 3.2.21. We define $\binom{H_i}{N} := \frac{H_i(H_i-1)\cdots(H_i-(N-1))}{N!}$.

Lemma 3.2.22. *We have*

$$\left\langle \binom{H_2}{N}, e^\mu \right\rangle = \delta_{0,\mu_a}\delta_{0,\mu_b}\delta_{0,\mu_c}\delta_{N,\mu_d}$$

and

$$\left\langle \binom{H_1}{N}, e^\mu \right\rangle = \delta_{N,\mu_a}\delta_{0,\mu_b}\delta_{0,\mu_c}\delta_{0,\mu_d}.$$

Proof. We show the first equation by induction on N. The second is obtained analogously.
We know that

$$\langle H_2, e^\mu \rangle = \delta_{0,\mu_a}\delta_{0,\mu_b}\delta_{0,\mu_c}\delta_{1,\mu_d}$$

by Lemma 3.2.6 and hence the first equation is true for $N = 1$. Now assume it is true for some $N \in \mathbb{N}$. Recall that in Lemma 3.2.7 we showed that

$$\langle H_2 X, e^\mu \rangle = (\mu_d + \mu_c)\langle X, e^\mu \rangle + \mu_d \langle X, e^\mu_{d,-1} \rangle.$$

Combining with $\binom{H_2}{N+1} = \frac{H_2 - N}{N+1}\binom{H_2}{N}$ we obtain

$$\begin{aligned}
\left\langle \binom{H_2}{N+1}, e^\mu \right\rangle &= \left\langle \frac{H_2 - N}{N+1}\binom{H_2}{N}, e^\mu \right\rangle \\
&= \frac{\mu_d + \mu_b - N}{N+1}\left\langle \binom{H_2}{N}, e^\mu \right\rangle + \frac{\mu_d}{N+1}\left\langle \binom{H_2}{N}, e^\mu_{d,-1} \right\rangle \\
&= \frac{\mu_d + \mu_b - N}{N+1}\delta_{0,\mu_a}\delta_{0,\mu_b}\delta_{0,\mu_c}\delta_{N,\mu_d} + \frac{\mu_d}{N+1}\delta_{0,\mu_a}\delta_{0,\mu_b}\delta_{0,\mu_c}\delta_{N,\mu_d - 1} \\
&= \delta_{0,\mu_a}\delta_{0,\mu_b}\delta_{0,\mu_c}\delta_{N+1,\mu_d}
\end{aligned}$$

The last equality is true because for $\mu_b = 0$, $\mu_d = N$ we have $\frac{\mu_d + \mu_b - N}{N+1} = 0$. $\square$

Corollary 3.2.23. *Let m be any (noncommutative) monomial in the variables $a-1, b, c, d-1$ and for $x \in \{a-1, b, c, d-1\}$ let μ_x be the number of times that x occurs in m. Then*

$$\begin{aligned}
\left\langle \binom{H_2}{N}, m \right\rangle &= \delta_{0,\mu_a}\delta_{0,\mu_b}\delta_{0,\mu_c}\delta_{N,\mu_d} \\
\left\langle \binom{H_1}{N}, m \right\rangle &= \delta_{N,\mu_a}\delta_{0,\mu_b}\delta_{0,\mu_c}\delta_{0,\mu_d}.
\end{aligned}$$

Proof. We will only show the first equation. Let $K[[t]]\langle a, b, c, d\rangle$ be the t-adic completion of $K[t]\{a, b, c, d\}$ and let $K[[t]]\langle d\rangle$ be the t-adic completion of

$K[t][d]$. Consider the $K[[t]]$-algebra morphism

$$\begin{aligned} K[[t]]\langle a,b,c,d\rangle &\longrightarrow K[[t]]\langle d\rangle \\ a &\longmapsto 1 \\ b &\longmapsto 0 \\ c &\longmapsto 0 \\ d &\longmapsto d. \end{aligned}$$

Since the relations defining $M_q(2,t)$ are in the kernel of this map we obtain a $K[[t]]$-algebra morphism

$$\phi : M_q(2,t) \longrightarrow K[[t]]\langle d\rangle.$$

Note that the set

$$\left\{e^\eta : \eta \in N_0^4 \text{ and } \eta_a = \eta_b = \eta_c = 0\right\}$$

is a $K[[t]]$-linearly independent in $K[[t]]\langle d\rangle$. We know that there are finitely many $a_\eta \in K[[t]]$ such that $m = \sum_\eta a_\eta e^\eta$. Now assume that $\mu_a \neq 0$ or $\mu_b \neq 0$ or $\mu_c \neq 0$. Then $\phi(m) = 0$. But since we also know that

$$\phi(m) = \phi\left(\sum_\eta a_\eta e^\eta\right) = \sum_{\eta_a=\eta_b=\eta_c=0} a_\eta e^\eta$$

we can conclude that

$$0 = \phi(m) = \sum_{\eta_a=\eta_b=\eta_c=0} a_\eta e^\eta$$

This implies that $a_\eta = 0$ for all η with $\eta_a = \eta_b = \eta_c = 0$. Thus if $\mu_a \neq 0$ or $\mu_b \neq 0$ or $\mu_c \neq 0$ then by Lemma 3.2.22 we know that

$$\begin{aligned} \left\langle \binom{H_2}{N}, m \right\rangle &= \left\langle \binom{H_2}{N}, \sum_\eta a_\eta e^\eta \right\rangle \\ &= \left\langle \binom{H_2}{N}, \sum_{\eta_a=\eta_b=\eta_c=0} a_\eta e^\eta \right\rangle \\ &= \left\langle \binom{H_2}{N}, 0 \right\rangle = 0 \end{aligned}$$

Since for $m = (d-1)^{\mu_d}$ Lemma 3.2.22 implies that $\left\langle \binom{H_2}{N}, (d-1)^{\mu_d} \right\rangle = \delta_{N,\mu_d}$ the first equation in the corollary is shown. $\square$

Lemma 3.2.24. *Let $N, n, m, M \in \mathbb{N}_0$ and let $\mu = (\mu_a, \mu_b, \mu_c, \mu_d) \in \mathbb{N}_0^4$. Then*

1. *If $n-\mu_b \neq m-\mu_b$ or $n<\mu_b$ or $\mu_a < n-\mu_b$ then*

$$\left\langle \binom{H_1}{N} F^n E^m \binom{H_2}{M}, e^\mu \right\rangle = 0.$$

2. *If $n \geq \mu_b$ then*

$$\left\langle \binom{H_1}{N} F^n E^m \binom{H_2}{M}, e^\mu \right\rangle$$

is an element of

$$t^{\max\{0,\mu_d - M\}} \frac{(q^2,q^2)_n (q^2,q^2)_m}{(1-q^2)^{n+m}} K[[t]]^\circ_{r_t}.$$

3. *If $n \geq \mu_b$ and $\mu_a - N \geq n - \mu_b \geq 0$ then*

$$\left\langle \binom{H_1}{N} F^n E^m \binom{H_2}{M}, e^\mu \right\rangle$$

is an element of

$$t^{\max\{0,\mu_d - M\}+\mu_a - N-(n-\mu_b)} \frac{(q^2,q^2)_n (q^2,q^2)_m}{(1-q^2)^{n+m}} K[[t]]^\circ_{r_t}.$$

4. *If $\mu = (N,n,m,M)$ then*

$$\left\langle \binom{H_1}{N} F^n E^m \binom{H_2}{M}, e^\mu \right\rangle = q^z \frac{(q^2,q^2)_n (q^2,q^2)_m}{(1-q^2)^{n+m}} (1+tf)$$

for some $z \in \mathbb{Z}$ and $f \in K[[t]]^\circ_{r_t}$.

Proof. Recall that

$$\begin{aligned}
\Delta(a-1) =& (a-1)\otimes(a-1) + b\otimes c + 1\otimes(a-1) + (a-1)\otimes 1\\
\Delta(b) =& (a-1)\otimes b + b\otimes(d-1) + b\otimes 1 + 1\otimes b\\
\Delta(c) =& c\otimes(a-1) + (d-1)\otimes c + c\otimes 1 + 1\otimes c\\
\Delta(d-1) =& (d-1)\otimes(d-1) + c\otimes b + 1\otimes(d-1) + (d-1)\otimes 1.
\end{aligned}$$

Thus we have a description of $\Delta(e^\mu)$ as a sum of elementary tensors. We will use the following property of the bracket:

$$\left\langle \binom{H_1}{N} F^n E^m \binom{H_2}{M}, e^\mu \right\rangle = \left\langle \binom{H_1}{N} \otimes F^n E^m \binom{H_2}{M}, \Delta(e^\mu) \right\rangle.$$

By Corollary 3.2.23 we only need to consider the elementary tensors in $\Delta(e^\mu)$ of the form $(a-1)^N \otimes m$. These are of the shape

$$(a-1)^N \otimes (a-1)^i b^{\mu_b} c^{\mu_c} (d-1)^{\mu_d}$$

where $\max\{0, \mu_a - N\} \leq i \leq \mu_a$. This can be seen by analyzing $\Delta(e^\mu)$ as in the proof of Lemma 3.2.7.
Because of

$$\begin{aligned}&\left\langle \binom{H_1}{N} \otimes F^n E^m \binom{H_2}{M}, (a-1)^N \otimes (a-1)^i b^{\mu_b} c^{\mu_c} (d-1)^{\mu_d} \right\rangle \\ &= \left\langle F^n E^m \binom{H_2}{M}, (a-1)^i b^{\mu_b} c^{\mu_c} (d-1)^{\mu_d} \right\rangle\end{aligned}$$

we only have to estimate the right hand side. Using the same strategy for H_2 we only have to estimate

$$\left\langle F^n E^m, (a-1)^i b^{\mu_b} c^{\mu_c} (d-1)^j \right\rangle$$

where $\max\{0, \mu_a - N\} \leq i \leq \mu_a$ and $\max\{0, \mu_d - M\} \leq j \leq \mu_d$. This is zero if $n - \mu_b \neq m - \mu_c$ or if $i < n - \mu_b$ or $n - \mu_b < 0$ by Lemma 3.2.20 and we can conclude the first statement of the Theorem.
If $i < n - \mu_b$ then Lemma 3.2.20 implies that

$$\left\langle F^n E^m, (a-1)^i b^{\mu_b} c^{\mu_c} (d-1)^j \right\rangle = 0. \tag{3.2.9}$$

If $i \geq n - \mu_b$ then

$$\left\langle F^n E^m, (a-1)^i b^{\mu_b} c^{\mu_c} (d-1)^j \right\rangle \in t^{j+i-(n-\mu_b)} \frac{(q^2, q^2)_n (q^2, q^2)_m}{(1-q^2)^{n+m}} K[[t]]^\circ_{r_t}. \tag{3.2.10}$$

Thus $\max\{0, \mu_a - N\} \leq i \leq \mu_a$ and $\max\{0, \mu_d - M\} \leq j \leq \mu_d$ imply the second and the third statement.
Let $\mu = (N, n, m, M)$. Then going along the lines of the arguments above for one term we have

$$\begin{aligned}&\left\langle \binom{H_1}{N} \otimes F^n E^m \otimes \binom{H_2}{M}, (a-1)^N \otimes b^{\mu_b} c^{\mu_c} \otimes (d-1)^{\mu_d} \right\rangle \\ &= \langle F^n E^m, b^{\mu_b} c^{\mu_c} \rangle = \frac{(q^2, q^2)_n (q^2, q^2)_m}{(1-q^2)^{n+m}}.\end{aligned}$$

Since for all the other terms $i + j > 0$ and $n - \mu_b = 0$ in (3.2.10) also the fourth statement is proven. □

Theorem 3.2.25. *Recall that on $K[[t]]_{r_t}$ we defined the valuation ν by*

$$\nu\left(\sum_{k \geq 0} a_k t^k\right) = \min\{\nu(a_k) + r_t k\}.$$

Let $\max\left\{4, \frac{2e}{p-1}\right\} \leq 2r < r_t$. *Then for* $\eta, \mu \in \mathbb{N}_0^4$ *and*

$$d_{\eta,\mu} := \left\langle \pi^{\lfloor r|\eta|\rfloor} \binom{H_1}{\eta_1} \frac{F^{\eta_2}}{\eta_2!} \frac{E^{\eta_3}}{\eta_3!} \binom{H_2}{\eta_4}, \pi^{-\lfloor r|\mu|\rfloor} e^{\mu} \right\rangle$$

we have that $d_{\eta,\mu} \in K[[t]]_{r_t}$ *and*

$$\nu\left(d_{\eta,\mu}\right) \geq \frac{1}{12} \lfloor r|\eta - \mu|\rfloor.$$

Moreover there exists $\epsilon > 0$ *such that*

$$\nu\left(d_{\eta,\mu}\right) \geq 1 + \epsilon$$

for all $\eta \neq \mu$.

Proof. Note that by Lemma 3.1.4 we know that $\nu(n!) = \nu\left(\frac{(q^2,q^2)_n}{(1-q^2)^n}\right)$ and thus when using Lemma 3.2.24 we can replace $\frac{(q^2,q^2)_n}{(1-q^2)^n}$ by $n!$. Then Lemma 3.2.24 implies that $d_{\eta,\mu} \in K[[t]]_{r_t}$. Moreover we will use without further mentioning them the inequalities

$$\lfloor x\rfloor - \lfloor y\rfloor \geq \lfloor x - y\rfloor \geq \lfloor x\rfloor - \lfloor y\rfloor - 1$$

and $\left\lfloor \frac{x}{3} \right\rfloor \geq \frac{1}{3}\lfloor x\rfloor - 1$ for $x, y \in \mathbb{R}$.
By Lemma 3.2.24 we only have to check the cases where $\eta_b - \mu_b = \eta_c - \mu_c \geq 0$ and $\mu_a \geq \eta_b - \mu_b$ which we will assume for the rest of the proof.
Case 1: Let $\eta_a \geq \mu_a$ and $\eta_d \geq \mu_d$. Then $|\eta| - |\mu| = |\eta - \mu|$ and thus

$$\lfloor r|\eta|\rfloor - \lfloor r|\mu|\rfloor \geq \lfloor r(|\eta| - |\mu|)\rfloor = \lfloor r|\eta - \mu|\rfloor.$$

Hence by Lemma 3.2.24/2 we know that

$$\nu\left(d_{\eta,\mu}\right) \geq \lfloor r|\eta|\rfloor - \lfloor r|\mu|\rfloor \geq \lfloor r|\eta - \mu|\rfloor$$

since $\nu(f) \geq 0$ for all $f \in K[[t]]_{r_t}^{\circ}$. If $\eta \neq \mu$ then $r \geq 2$ this implies that $\nu\left(d_{\eta,\mu}\right) \geq 2$.
Case 2 Let $\eta_a \geq \mu_a$ and $\eta_d < \mu_d$. Then $|\eta - \mu| = |\eta| - |\mu| + 2(\mu_d - \eta_d)$ and hence

$$\lfloor r|\eta|\rfloor - \lfloor r|\mu|\rfloor \geq \lfloor r|\eta - \mu| - 2r(\mu_d - \eta_d)\rfloor \geq \lfloor r|\eta - \mu|\rfloor - \lfloor 2r(\mu_d - \eta_d)\rfloor - 1.$$

Thus by Lemma 3.2.24/2 we have

$$\begin{aligned}
\nu\left(d_{\eta,\mu}\right) &\geq \lfloor r|\eta|\rfloor - \lfloor r|\mu|\rfloor + r_t(\mu_d - \eta_d) \\
&\geq \lfloor r|\eta-\mu|\rfloor - \lfloor 2r(\mu_d - \eta_d)\rfloor - 1 + r_t(\mu_d - \eta_d) \\
&\geq \lfloor r|\eta-\mu|\rfloor - 1 + r_t(\mu_d - \eta_d) - \lfloor 2r(\mu_d - \eta_d)\rfloor \\
&\geq \frac{1}{2}\lfloor r|\eta-\mu|\rfloor + r_t(\mu_d - \eta_d) - \lfloor 2r(\mu_d - \eta_d)\rfloor \\
&\geq \frac{1}{2}\lfloor r|\eta-\mu|\rfloor
\end{aligned}$$

since $r \geq 2$. Moreover because of $r_t(\mu_d - \eta_d) - \lfloor 2r(\mu_d - \eta_d)\rfloor \geq r_t - 2r$ and $r \geq 2$ we obtain $\nu\left(d_{\eta,\mu}\right) \geq 1 + r_t - 2r$.

Case 3: Let $\eta_a < \mu_a$ and $\eta_d < \mu_d$. Assume furthermore that $\mu_a - \eta_a \leq \eta_b - \mu_b$. Then

$$\begin{aligned}
|\eta| - |\mu| &= \eta_a - \mu_a + 2(\eta_b - \mu_b) + \eta_d - \mu_d \\
&\geq \eta_b - \mu_b + \mu_d - \eta_d - 2(\mu_d - \eta_d) \\
&\geq \frac{1}{3}|\eta - \mu| - 2(\mu_d - \eta_d).
\end{aligned}$$

and thus similar to Case 2 we have

$$\begin{aligned}
\lfloor r|\eta|\rfloor - \lfloor r|\mu|\rfloor &\geq \left\lfloor \frac{1}{3} r|\eta-\mu| - 2r(\mu_d - \eta_d)\right\rfloor \\
&\geq \frac{1}{3}\lfloor r|\eta-\mu|\rfloor - \lfloor 2r(\mu_d - \eta_d)\rfloor - 2.
\end{aligned}$$

Thus

$$\begin{aligned}
\nu\left(d_{\eta,\mu}\right) &\geq \lfloor r|\eta|\rfloor - \lfloor r|\mu|\rfloor + r_t(\mu_d - \eta_d) \\
&\geq \frac{1}{3}\lfloor r|\eta-\mu|\rfloor - \lfloor 2r(\mu_d - \eta_d)\rfloor - 2 + r_t(\mu_d - \eta_d) \\
&\geq \frac{1}{3}\lfloor r|\eta-\mu|\rfloor - 2 \\
&\geq \frac{1}{12}\lfloor r|\eta-\mu|\rfloor
\end{aligned}$$

since $|\eta - \mu| \geq 4$ in this case and thus $\frac{1}{4}\lfloor r|\eta-\mu|\rfloor \geq 2$. We also have that

$$|\eta| - |\mu| = \eta_a - \mu_a + 2(\eta_b - \mu_b) + \eta_d - \mu_d \geq 1 - 2(\mu_d - \eta_d).$$

Concluding as above we obtain

$$\begin{aligned}
\nu\left(d_{\eta,\mu}\right) &\geq \lfloor r|\eta|\rfloor - \lfloor r|\mu|\rfloor + r_t(\mu_d - \eta_d) \\
&\geq \lfloor r(1 - 2(\mu_d - \eta_d))\rfloor + r_t(\mu_d - \eta_d) \\
&\geq \lfloor r\rfloor - \lfloor 2r(\mu_d - \eta_d)\rfloor - 1 + r_t(\mu_d - \eta_d) \\
&\geq 1 + r_t(\mu_d - \eta_d) - \lfloor 2r(\mu_d - \eta_d)\rfloor \\
&\geq 1 + r_t - 2r
\end{aligned}$$

Case 4: Let $\eta_a < \mu_a$ and $\eta_d \geq \mu_d$. Assume furthermore that $\mu_a - \eta_a \leq \eta_b - \mu_b$. Similar as in Case 3 on shows that

$$\lfloor r|\eta|\rfloor - \lfloor r|\mu|\rfloor \geq \frac{1}{3}\lfloor r|\eta - \mu|\rfloor - 1$$

and concludes with Lemma 3.2.24/2 and $|\eta - \mu| \geq 3$ that

$$\begin{aligned}
\nu\left(d_{\eta,\mu}\right) &\geq \lfloor r|\eta|\rfloor - \lfloor r|\mu|\rfloor \\
&\geq \frac{1}{3}\lfloor r|\eta - \mu|\rfloor - 1 \\
&\geq \frac{1}{6}\lfloor r|\eta - \mu|\rfloor
\end{aligned}$$

Moreover in this case we have that

$$|\eta| - |\mu| = \eta_a - \mu_a + 2(\eta_b - \mu_b) + \eta_d - \mu_d \geq \eta_b - \mu_b \geq 1$$

and thus

$$\nu\left(d_{\eta,\mu}\right) \geq \lfloor r|\eta|\rfloor - \lfloor r|\mu|\rfloor \geq \lfloor r\rfloor \geq 2.$$

Case 5: Let $\eta_a < \mu_a$, $\eta_d < \mu_d$ and $\eta_b = \mu_b$. Then

$$|\eta - \mu| = \mu_a - \eta_a + \mu_d - \eta_d = |\eta| - |\mu| - 2(\mu_a - \eta_a + \mu_d - \eta_d)$$

and thus

$$\lfloor r|\eta|\rfloor - \lfloor r|\mu|\rfloor \geq \lfloor r|\eta - \mu|\rfloor - \lfloor 2r(\mu_a - \eta_a + \mu_d - \eta_d)\rfloor - 1$$

and hence by Lemma 3.2.24/3

$$\begin{aligned}
\nu\left(d_{\eta,\mu}\right) &\geq \lfloor r|\eta|\rfloor - \lfloor r|\mu|\rfloor + r_t(\mu_a - \eta_a + \mu_d - \eta_d) \\
&\geq \lfloor r|\eta - \mu|\rfloor - \lfloor 2r(\mu_a - \eta_a + \mu_d - \eta_d)\rfloor - 1 + r_t(\mu_a - \eta_a + \mu_d - \eta_d) \\
&\geq \frac{1}{2}\lfloor r|\eta - \mu|\rfloor + r_t - 2r \\
&\geq \frac{1}{2}\lfloor r|\eta - \mu|\rfloor.
\end{aligned}$$

Consequently $\nu\left(d_{\eta,\mu}\right) \geq 1 + r_t - 2r$.

Case 6: Let $\eta_a < \mu_a$, $\eta_d \geq \mu_d$ and $\eta_b = \mu_b$.
As in Case 5 we obtain $\nu\left(d_{\eta,\mu}\right) \geq \frac{1}{2}\lfloor r|\eta-\mu|\rfloor$ and $\nu\left(d_{\eta,\mu}\right) \geq 1 + r_t - 2r$.
Case 7: Let $\eta_a < \mu_a$, $\eta_d < \mu_d$ and $\mu_a - \eta_a > \eta_b - \mu_b > 0$. Then

$$\begin{aligned}|\eta| - |\mu| &= \eta_a - \mu_a + 2(\eta_b - \mu_b) + \eta_d - \mu_d \\ &\geq \mu_a - \eta_a + \mu_d - \eta_d - 2(\mu_a - \eta_a - (\eta_b - \mu_b) + \mu_d - \eta_d) \\ &\geq \frac{1}{3}|\eta - \mu| - 2(\mu_a - \eta_a - (\eta_b - \mu_b) + \mu_d - \eta_d). \qquad (3.2.11)\end{aligned}$$

and thus

$$\begin{aligned}\lfloor r|\eta|\rfloor - \lfloor r|\mu|\rfloor &\geq \left\lfloor \frac{1}{3} r|\eta-\mu| - 2r(\mu_a - \eta_a - (\eta_b - \mu_b) + \mu_d - \eta_d)\right\rfloor \\ &\geq \frac{1}{3}\lfloor r|\eta-\mu|\rfloor - \lfloor 2r(\mu_a - \eta_a - (\eta_b - \mu_b) + \mu_d - \eta_d)\rfloor - 2.\end{aligned}$$

Hence using $|\eta - \mu| \geq 4$ and Lemma 3.2.24/3

$$\begin{aligned}\nu\left(d_{\eta,\mu}\right) &\geq \lfloor r|\eta|\rfloor - \lfloor r|\mu|\rfloor + r_t(\mu_a - \eta_a - (\eta_b - \mu_b) + \mu_d - \eta_d) \\ &\geq \frac{1}{3}\lfloor r|\eta-\mu|\rfloor - \lfloor 2r(\mu_a - \eta_a - (\eta_b - \mu_b) + \mu_d - \eta_d)\rfloor - 2 \\ &+ r_t(\mu_a - \eta_a - (\eta_b - \mu_b) + \mu_d - \eta_d) \\ &\geq \frac{1}{3}\lfloor r|\eta-\mu|\rfloor - 2 \\ &\geq \frac{1}{12}\lfloor r|\eta-\mu|\rfloor.\end{aligned}$$

Using (3.2.11) and $|\eta - \mu| \geq 4$ we can conclude that

$$|\eta| - |\mu| \geq 1 - 2(\mu_a - \eta_a - (\eta_b - \mu_b) + \mu_d - \eta_d)$$

and thus $\lfloor r|\eta|\rfloor - \lfloor r|\mu|\rfloor \geq \lfloor r\rfloor - \lfloor 2r(\mu_a - \eta_a - (\eta_b - \mu_b) + \mu_d - \eta_d)\rfloor - 1$. Using $r \geq 2$ and Lemma 3.2.24/3 we obtain

$$\begin{aligned}\nu\left(d_{\eta,\mu}\right) &\geq \lfloor r|\eta|\rfloor - \lfloor r|\mu|\rfloor + r_t(\mu_a - \eta_a - (\eta_b - \mu_b) + \mu_d - \eta_d) \\ &\geq 2 - 1 - \lfloor 2r(\mu_a - \eta_a - (\eta_b - \mu_b) + \mu_d - \eta_d)\rfloor \\ &+ r_t(\mu_a - \eta_a - (\eta_b - \mu_b) + \mu_d - \eta_d) \\ &\geq 1 + r_t - 2r.\end{aligned}$$

Case 8: Let $\eta_a < \mu_a$, $\eta_d \geq \mu_d$ and $\mu_a - \eta_a > \eta_b - \mu_b > 0$. Similar as in Case 7 we obtain $\nu\left(d_{\eta,\mu}\right) \geq \frac{1}{12}\lfloor r|\eta-\mu|\rfloor$ and $\nu\left(d_{\eta,\mu}\right) \geq 1 + r_t - 2r$.

Thus we proved the first statement and the second statement with

$$\epsilon = \min\{2, r_t - 2r\} > 0.$$

$\square$

3.2.3 Divided power subalgebras of U_t

In this section we will introduce a K-Banach algebra that is closely related to the partial divided power algebras which will be used to show that $D_q(H,K)$ is a Fréchet Stein algebra.

Definition 3.2.26. Let $n \in \mathbb{N}_0, m \in \mathbb{N}$. We define $l_n^m \in \mathbb{N}_0$ to be the unique number such that

$$n = l_n^m p^m + r$$

with $0 \leq r < p^m$. Furthermore we define $\gamma_n^m := \frac{l_n^m!}{n!}$.

3.2.27. The set $\{\gamma_n^m : n \in \mathbb{N}_0\}$ defines a system of partial divided powers for fixed $m \in \mathbb{N}$. Partial divided powers were introduced by P. Berthelot in [Ber96] section 1.3. The usage in this context is due to M. Emerton, see [Eme11] section 5.2.
Let $R := \left\{\sum_i a_i \frac{x^i}{i!} : a_i \in \mathcal{O} \text{ and } \lim a_i = 0\right\} \subseteq K[[x]]$ with valuation

$$\nu_R\left(\sum_i a_i \frac{x^i}{i!}\right) = \min\{\nu(a_i)\}.$$

We have that $\nu(n!m!) < \nu((n+m)!)$ for many $m, n \in \mathbb{N}$ and thus in general there exists no $a \in \mathcal{O}^\times$ with $\frac{X^n}{n!}\frac{X^m}{m!} = a\frac{X^{n+m}}{(m+n)!}$. As a consequence the graded ring for the filtration induced by ν_R is not finitely generated as $\mathcal{O}/\pi$-algebra. This problem can be solved by using partial divided powers. Our usage of partial divided powers in section **3.3** will be very similar.

Definition 3.2.28. We define for $\mu \in \mathbb{N}_0^4$

$$D_\mu := \frac{H_1^{\mu_1} F^{\mu_2} E^{\mu_3} H_2^{\mu_4}}{\mu_1!\mu_2!\mu_3!\mu_4!}$$

$$\bar{D}_\mu := \binom{H_1}{\mu_1}\frac{F^{\mu_2}E^{\mu_3}}{\mu_2!\mu_3!}\binom{H_2}{\mu_4}$$

$$D_\mu^m := \gamma_\mu^m H_1^{\mu_1} F^{\mu_2} E^{\mu_3} H_2^{\mu_4}.$$

We fix an element $r_t \in \mathbb{R}$, $r_t > \max\left\{4, \frac{6}{p-1}\right\}$ which will correspond to the valuation of t and which will later be the valuation of $1-q$. For the definition of $K[[t]]_{r_t}$ see **3.2.16**. We consider the following $K[[t]]_{r_t}$-submodules of U_t

$$U_t^b(r, r_t) := \left\{ \sum_\mu a_\mu D_\mu \in U_t : \nu(a_\mu) - r|\mu| \to \infty \text{ for } |\mu| \to \infty \right\}$$
$$U_t^{an}(r, r_t) := \left\{ \sum_\mu a_\mu \bar{D}_\mu \in U_t : \nu(a_\mu) - r|\mu| \text{ is bounded from below} \right\}$$
$$U_t^m(r, r_t) := \left\{ \sum_\mu a_\mu D_\mu^m \in U_t : \nu(a_\mu) - r|\mu| \to \infty \text{ for } |\mu| \to \infty \right\}$$

where the coefficients a_μ are elements of $K[[t]]_{r_t}$. As we will see later these submodules are subalgebras of U_t.
We have natural valuations on them given by

$$\nu_{U_t^b(r,r_t)} \left(\sum_\mu a_\mu D_\mu \right) = \min\{\nu(a_\mu) - r|\mu|\}$$
$$\nu_{U_t^{an}(r,r_t)} \left(\sum_\mu a_\mu \bar{D}_\mu \right) = \inf\{\nu(a_\mu) - r|\mu|\}$$
$$\nu_{U_t^m(r,r_t)} \left(\sum_\mu a_\mu D_\mu^m \right) = \min\{\nu(a_\mu) - r|\mu|\}.$$

The corresponding spaces are not complete with respect to these valuations. However the quotients we will consider in order to describe the distribution algebras will be.

3.2.29. The valuations of the previous definition are well defined. We will show this for $U_t^{an}(r, r_t)$. We have to show that the presentation as sum is unique i.e. that for

$$f = \sum_\mu a_\mu \bar{D}_\mu \in U_t$$

with $a := \inf\{\nu(a_\mu) - r|\mu|\} > -\infty$ we have that $f = 0$ iff $a_\mu = 0$ for all $\mu \in \mathbb{N}_0^4$. Assume that $\inf\{\nu(a_\mu) - r|\mu|\} < \infty$. Let $\epsilon > 0$ be as in Theorem 3.2.25. Let $\alpha \in \mathbb{N}_0^4$ be such that $\nu(a_\alpha) - r|\alpha| \le a + \frac{\epsilon}{2}$. Then by Theorem 3.2.25 we have that $\nu\left(\langle a_\alpha \bar{D}_\alpha, e^\alpha \rangle\right) = \nu(a_\alpha) \le a + \frac{\epsilon}{2} + r|\alpha|$. For $\eta \neq \alpha$ we know by Theorem 3.2.25 that

$$\begin{aligned} \nu\left(\langle a_\eta \bar{D}_\eta, e^\alpha \rangle\right) &\ge \nu(a_\eta) - \lfloor r|\eta| \rfloor + \lfloor r|\alpha| \rfloor + 1 + \epsilon \\ &\ge \nu(a_\eta) - r|\eta| + r|\alpha| - 1 + 1 + \epsilon \\ &\ge a + r|\alpha| + \epsilon > \nu\left(\langle a_\alpha \bar{D}_\alpha, e^\alpha \rangle\right) \end{aligned}$$

Since the bracket is t-adically continuous we can conclude that

$$\nu\left(\left\langle\sum_{\mu} a_\mu \bar{D}_\mu, e^\alpha\right\rangle\right) = \nu\left(\left\langle a_\alpha \bar{D}_\alpha, e^\alpha\right\rangle\right) \leq a + \frac{\epsilon}{2} + r|\alpha| < \infty$$

and thus $\sum_\mu a_\mu \bar{D}_\mu \neq 0$

Remark 3.2.30. For the investigation of the distribution algebras that we will consider later, we will need to consider modules obtained by replacing t by a sufficiently small $h \in \mathcal{O}$ in the spaces $U_t^b(r, r_t), U_t^{an}(r, r_t), U_t^m(r, r_t)$. Since we want to see that the module we obtain from $U_t^m(r, r_t)$ by this procedure is an algebra, we will show the corresponding for the t-adic space. Therefore we need some auxiliary computations.

Lemma 3.2.31. *Recall that $K_i = e^{tH_i}$ and let $K := K_1^{-1}K_2$.*

1. *$K_1, K_2, K_1^{-1}K_2^{-1} \in U$ for U one of the modules in Definition 3.2.26.*
2. *For $n \in \mathbb{Z}$ let $[K, n] := \frac{Kq^n - K^{-1}q^{-n}}{q - q^{-1}}$. Then*

$$[K, n] = a + \sum_{i+j\geq 1} a_{i,j} t^{i+j-1} \frac{H_1^i H_2^j}{i!j!}$$

for some $a, a_{i,j} \in K[[t]]_{r_t}^\circ$.

3. *For $n \in \mathbb{Z}, k \in \mathbb{N}$ let $[K, n, k] = [K, n] \cdots [K, n-k+1]$ and let furthermore $[K, n, 0] = 1$. Then*

$$[K, n, k] = \sum_{i+j<k} a_{i,j} \frac{H_1^i H_2^j}{i!j!} + \sum_{i+j\geq k} a_{i,j} t^{i+j-k} \frac{H_1^i H_2^j}{i!j!}$$

for some $a_{i,j} \in K[[t]]_{r_t}^\circ$.

4. *$E^n F^m = \sum_{k=0}^{\min\{n,m\}} \frac{[n]![m]!}{[k]![n-k]![m-k]!} F^{m-k} E^{n-k} [K, n-m, k]$.*

Proof. 1. Clear.

2. We can write

$$\begin{aligned}[K, n] &= \frac{\exp(-t(H_1 - H_2 - n)) - \exp(t(H_1 - H_2 - n))}{\exp(t) - \exp(-t)} \\ &= -\frac{\sum_{k\geq 0} \frac{t^{2k+1}(H_1 - H_2 - n)^{2k+1}}{(2k+1)!}}{\sum_{k\geq 0} \frac{t^{2k+1}}{(2k+1)!}}.\end{aligned}$$

The denominator can be written as tu with $u \in \left(K[[t]]_{r_t}^\circ\right)^\times$. Hence the equation follows with the binomial formula.

3. Follows from 2.

4. [KS97] section 3.1.2 Proposition 5.

□

Lemma 3.2.32. *Consider the following equations on U_t.*

$$\begin{aligned}
\gamma_n^m H_2^n \gamma_j^m F^j &= \gamma_j^m F^j \gamma_n^m H_2^n + \sum_{\mu \neq (0,j,0,n)} a_\mu D_\mu^{t,m};\\
\gamma_n^m H_2^n \gamma_j^m E^j &= \gamma_j^m E^j \gamma_n^m H_2^n + \sum_{\mu \neq (0,0,j,n)} a_\mu D_\mu^{m}\\
\gamma_j^m F^j \gamma_n^m H_1^n &= \gamma_n^m H_1^n \gamma_j^m F^j + \sum_{\mu \neq (n,j,0,0)} a_\mu D_\mu^{m};\\
\gamma_j^m E^j \gamma_n^m H_1^n &= \gamma_n^m H_1^n \gamma_j^m E^j + \sum_{\mu \neq (n,0,j,0)} a_\mu D_\mu^{m}.
\end{aligned}$$

Recall that $e = \nu(p)$ and let $\max\left\{4, \frac{6e}{p-1}\right\} \leq 2r < r_t$. Then in either of these cases

$$\nu_{U_t^m(r,r_t)}\left(a_\mu D_\mu^m\right) > -r(n+j).$$

Moreover there exists $c > 0$ independent of n, j such that

$$\nu_{U_t^m(r,r_t)}\left(a_\mu D_\mu^m\right) \geq -r(n+j) + c|(n+j) - |\mu||.$$

Proof. We will only show the claim for the first expression, the other ones are proven similarly. Note that $H_2^n F^j = F^j (H_2 - j)^n$. Hence using the binomial formula we see that there exists a $a_k \in \mathbb{Z}$ such that

$$\gamma_n^m H_2^n \gamma_j^m F^j = \sum_{k \leq n} a_k \binom{n}{k} \frac{l_j^m! F^j}{j!} \frac{l_n^m! H_2^k}{n!} = \sum_{k \leq n} a_k \frac{l_n^m!}{l_k^m!(n-k)!} \gamma_j^m F^j \gamma_k^m H_2^k.$$

Recall that for $z \in \mathbb{N}_0$ we defined $S(z)$ to be the sum of the p-adic digits. Using that $\nu(z!) = e\frac{z - S(z)}{p-1}$ for $z \in \mathbb{N}_0$ and $\nu(l_n^m!) \geq \nu(l_k^m!)$ for $n \geq k$ and the assumption $r \geq \frac{3e}{p-1}$ we obtain a lower bound for the valuation of the summands of the right hand side as

$$-r(j+k) - \frac{e(n-k)}{p-1} = -r(n+j) - \frac{e(n-k)}{p-1} + r(n-k) \geq -r(n+j) + \frac{2e}{p-1}(n-k).$$

Since in this case $\mu = (0, j, 0, k)$ we have

$$|(n+j) - |\mu|| = n - k$$

and the claim for the first expression follows. □

Lemma 3.2.33. *Let $m_1, m_2, m_3 \in \mathbb{N}$, $r_1, r_2, r_3 \in \mathbb{R}$ and $j, n \in \mathbb{N}_0$. By Lemma*

3.2.31 there are $a_{k,\tau,\sigma} \in K[[t]]_{r_t}$ such that

$$\gamma_n^{m_2} E^n \gamma_j^{m_1} F^j = \gamma_j^{m_1} F^j \gamma_n^{m_2} E^n + \sum_{k=1}^{\min\{j,n\}} \sum_{\tau,\sigma} a_{k,\tau,\sigma} \gamma_{j-k}^{m_1} F^{j-k} \gamma_{n-k}^{m_2} E^{n-k} \gamma_\tau^{m_3} \gamma_\sigma^{m_3} H_1^\tau H_2^\sigma.$$

For $\alpha \in \mathbb{Z}^4$ define $|\alpha| = \sum_i |\alpha_i|$. Assume that

$$\max\left\{4, \frac{6e}{p-1}\right\} \leq 2r_1, 2r_2, 2r_3 < r_t$$

and that $r_3 \leq \max\{r_1, r_2\} + \frac{e}{p-1}$. Then there exists $c > 0$ independent of n, j such that

$$\nu(a_{k,\tau,\sigma}) - r_1(j-k) - r_2(n-k) - r_3(\tau+\sigma) \geq -r_1 j - r_2 n + c|(\tau, -k, -k, \sigma)|$$

where we write ν for $\nu_{U_t^m(r,r_t)}$. Assume in addition that $r_1 + r_2 - r_3 - \frac{e}{p-1} \geq 2$ and that $m_3 \geq 3$, then we obtain

$$\nu(a_{k,\tau,\sigma}) - r_1(j-k) - r_2(n-k) - r_3(\tau+\sigma) \geq -r_1 j - r_2 n + 2.$$

Proof. By Lemma 3.2.31.4/5 we know that we can write $\gamma_n^{m_2} E^n \gamma_j^{m_1} F^j$ as sum

$$\gamma_n^{m_2} E^n \gamma_j^{m_1} F^j = \gamma_j^{m_1} F^j \gamma_n^{m_2} E^n + \sum_{k=1}^{\min\{j,n\}} \sum_{\tau,\sigma} \alpha_{k,\tau,\sigma} F^{j-k} E^{n-k} H_1^\tau H_2^\sigma. \tag{3.2.12}$$

In order obtain the estimates in the first statement of the Lemma we distinguish two cases.
Case 1: Let $\sigma + \tau \geq k$.
Then by Lemma 3.2.31 the elements $\alpha_{k,\tau,\sigma} F^{j-k} E^{n-k} H_1^\tau H_2^\sigma$ are of the form

$$b \frac{l_n^{m_2}! l_j^{m_1}! t^{\sigma+\tau-k}}{n! j! \tau! \sigma!} \frac{[n]![j]!}{[k]![n-k]![j-k]!} F^{j-k} E^{n-k} H_1^\tau H_2^\sigma.$$

Here $b \in K[[t]]_b^\circ$ depends on n, j, k, σ, τ. Using $\nu([i]) = \nu(i)$ and the definition of γ_α^m these summands can be written as

$$a \frac{l_n^{m_2}! l_j^{m_1}! t^{\sigma+\tau-k}}{l_{n-k}^{m_2}! l_{j-k}^{m_1}! k! l_\tau^{m_3}! l_\sigma^{m_3}!} \gamma_{j-k}^{m_1} F^{j-k} \gamma_{n-k}^{m_2} E^{n-k} \gamma_\tau^{m_3} H_1^\tau \gamma_\sigma^{m_3} H_2^\sigma \tag{3.2.13}$$

for some $a \in K[[t]]_{r_t}^\circ$. Without loss of generality assume that $r_2 \geq r_1$ and thus by assumption $r_3 \leq r_2 + \frac{e}{p-1}$. Let

$$A := \nu\left(a\frac{l_n^{m_2}!l_j^{m_1}!t^{\sigma+\tau-k}}{l_{n-k}^{m_2}!l_{j-k}^{m_1}!k!l_\tau^{m_3}!l_\sigma^{m_3}!}\right) - r_1(j-k) - r_2(n-k) - r_3(\tau+\sigma).$$

Recall that $\nu(z!) = \frac{e(z-S(z))}{p-1}$ for $S(z)$ the sum of the p-adic digits of z. Furthermore $l_z^m = \left\lfloor \frac{z}{p^m} \right\rfloor$ and $\nu\left(\frac{l_n^{m_2}!l_j^{m_1}!}{l_{n-k}^{m_2}!l_{j-k}^{m_1}!}\right) \geq 0$. This implies that

$$\begin{aligned} A \geq & -\frac{e}{p-1}\left(k - S(k) + \left\lfloor\frac{\tau}{p^{m_3}}\right\rfloor + \left\lfloor\frac{\sigma}{p^{m_3}}\right\rfloor\right) + r_t(\tau+\sigma-k) \\ & - r_1(j-k) - r_2(n-k) - r_3(\tau+\sigma) \end{aligned}$$

Since $k \leq \sigma+\tau$ and $m_3 \geq 1$ and $p \geq 2$ we know that

$$k - S(k) + \left\lfloor\frac{\tau}{p^{m_3}}\right\rfloor + \left\lfloor\frac{\sigma}{p^{m_3}}\right\rfloor \leq \frac{3}{2}(\tau+\sigma)$$

and hence

$$\begin{aligned} A \geq & -\frac{3e}{2(p-1)}(\tau+\sigma) + \frac{2e}{p-1}(\tau+\sigma-k) + \left(r_t - \frac{2e}{p-1}\right)(\tau+\sigma-k) \\ & + r_1k + r_2k - r_3(\tau+\sigma) - r_1j - r_2n. \end{aligned}$$

Using $r_1 \geq \frac{3e}{p-1}$ and hence $r_1k \geq \frac{3e}{p-1}k$ we obtain

$$\begin{aligned} A \geq & -\frac{3e}{2(p-1)}(\tau+\sigma) + \frac{2e}{p-1}(\tau+\sigma-k) + \frac{3e}{p-1}k + r_2k \\ & + \left(r_t - \frac{2e}{p-1} - r_3\right)(\tau+\sigma-k) + r_3(\tau+\sigma-k) - r_3(\tau+\sigma) - r_1j - r_2n \\ = & \frac{e}{2(p-1)}(\tau+\sigma) + \frac{e}{p-1}k + r_2k + \left(r_t - \frac{2e}{p-1} - r_3\right)(\tau+\sigma-k) \\ & - r_3k - r_1j - r_2n \end{aligned}$$

In the proof we assumed that $r_2 \geq r_1$ and thus $0 \leq r_2 + \frac{e}{p-1} - r_3$by the assumption $r_3 \leq \max\{r_1, r_2\} + \frac{e}{p-1}$. Using also that

$$\left(r_t - \frac{2e}{p-1} - r_3\right)(\tau+\sigma-k) \geq 0$$

we get

$$\begin{aligned} A & \geq \frac{e}{2(p-1)}(\tau+\sigma) + \left(r_2 + \frac{e}{p-1} - r_3\right)k - r_1j - r_2n \\ & \geq \frac{e}{2(p-1)}(\tau+\sigma) - r_1j - r_2n. \end{aligned} \tag{3.2.14}$$

Since $|(\tau, -k, -k, \sigma)| = 2k + \tau + \sigma$ and we assumed $k \leq \tau + \sigma$ the inequality in the Lemma is fulfilled for $c = \frac{e}{6(p-1)}$. If $r_2 \leq r_1$ we can interchange the roles of r_1 and r_2 in the estimation and get the same result.

Case 2: Let $\sigma + \tau < k$.

Similarly to Case 1 we can use Lemma 3.2.31.4/5 to show that the summands in (3.2.12) have the form

$$a \frac{l_n^{m_2}! l_j^{m_1}!}{l_{n-k}^{m_2}! l_{j-k}^{m_1}! k! l_\tau^{m_3}! l_\sigma^{m_3}!} \gamma_{j-k}^{m_1} F^{j-k} \gamma_{n-k}^{m_2} E^{n-k} \gamma_\tau^{m_3} H_1^\tau \gamma_\sigma^{m_3} H_2^\sigma \tag{3.2.15}$$

for some $a \in K[[t]]_{r_t}^\circ$. We again assume that $r_2 \geq r_1$ and thus by assumption $0 \leq r_2 + \frac{e}{p-1} - r_3$. Similarly to Case 1 one can show that

$$k - S(k) + \left\lfloor \frac{\tau}{p^{m_3}} \right\rfloor + \left\lfloor \frac{\sigma}{p^{m_3}} \right\rfloor \leq \frac{3}{2} k.$$

Using also $r_1 \geq \frac{3e}{p-1}$ we can estimate

$$\begin{aligned}
&\nu \left(a \frac{l_n^{m_2}! l_j^{m_1}!}{l_{n-k}^{m_2}! l_{j-k}^{m_1}! k! l_\tau^{m_3}! l_\sigma^{m_3}!} \right) - r_1(j-k) - r_2(n-k) - r_3(\tau + \sigma) \\
&\geq -\frac{e}{p-1} \left(k - S(k) + \left\lfloor \frac{\tau}{p^{m_3}} \right\rfloor + \left\lfloor \frac{\sigma}{p^{m_3}} \right\rfloor \right) - r_1(j-k) - r_2(n-k) \\
&- r_3(\tau + \sigma) \\
&\geq -\frac{3e}{2(p-1)} k + \frac{3e}{p-1} k + r_2 k - r_1 j - r_2 n - r_3 k \\
&\geq \frac{e}{2(p-1)} k + \left(r_2 + \frac{e}{p-1} - r_3 \right) k - r_1 j - r_2 n \\
&\geq \frac{e}{2(p-1)} k - r_1 j - r_2 n.
\end{aligned} \tag{3.2.16}$$

Since again $|(\tau, -k, -k, \sigma)| = 2k + \tau + \sigma$ and since $k > \tau + \sigma$ the inequality in the Lemma is fulfilled for $c = \frac{e}{6(p-1)}$.

To show the second claim we also distinguish the two cases $\tau + \sigma \geq k$ and $\tau + \sigma < k$. Thus the summands we will investigate in the two cases are the same as in the two cases we discussed above.

Case 1: Let $\tau + \sigma \geq k$.

Then $\tau + \sigma \geq k \geq 1$. If $\tau + \sigma \geq 4p$ then the estimate (3.2.14) implies the claim. If $1 \leq \tau + \sigma < 4p$ then as a consequence of $m_3 \geq 3$ we obtain

$\left\lfloor \frac{\tau}{p^{m_3}} \right\rfloor + \left\lfloor \frac{\sigma}{p^{m_3}} \right\rfloor = 0$. Thus $r_1 + r_2 - r_3 - \frac{e}{p-1} \geq 2$ implies

$$\begin{aligned}
&\nu\left(a\frac{l_n^{m_2}!l_j^{m_1}!t^{\sigma+\tau-k}}{l_{n-k}^{m_2}!l_{j-k}^{m_1}!k!l_\tau^{m_3}!l_\sigma^{m_3}!}\right) - r_1(j-k) - r_2(n-k) - r_3(\tau+\sigma) \\
&\geq -\frac{e}{p-1}\left(k - S(k) + \left\lfloor \frac{\tau}{p^{m_3}} \right\rfloor + \left\lfloor \frac{\sigma}{p^{m_3}} \right\rfloor\right) + r_t(\tau+\sigma-k) \\
&- r_1(j-k) - r_2(n-k) - r_3(\tau+\sigma) \\
&\geq -\frac{e}{p-1}k + r_t(\tau+\sigma-k) + r_1k + r_2k - r_3(\tau+\sigma) - r_1j - r_2n \\
&\geq \left(r_1 + r_2 - r_3 - \frac{e}{p-1}\right)(\tau+\sigma) + (r_t - r_1 - r_2)(\tau+\sigma-k) - r_1j - r_2n \\
&\geq 2 - r_1j - r_2n
\end{aligned}$$

Case 2: Let $\tau + \sigma < k$.
Then for $k \geq 4p$ the estimation follows from (3.2.16). If $\tau + \sigma < k < 4p$ then $\left\lfloor \frac{\tau}{p^{m_3}} \right\rfloor + \left\lfloor \frac{\sigma}{p^{m_3}} \right\rfloor = 0$ and hence

$$\begin{aligned}
&\nu\left(a\frac{l_n^{m_2}!l_j^{m_1}!}{l_{n-k}^{m_2}!l_{j-k}^{m_1}!k!l_\tau^{m_3}!l_\sigma^{m_3}!}\right) - r_1(j-k) - r_2(n-k) - r_3(\tau+\sigma) \\
&\geq -\frac{e}{p-1}\left(k - S(k) + \left\lfloor \frac{\tau}{p^{m_3}} \right\rfloor + \left\lfloor \frac{\sigma}{p^{m_3}} \right\rfloor\right) - r_1(j-k) - r_2(n-k) \\
&- r_3(\tau+\sigma) \\
&\geq -\frac{e}{p-1}k + r_1k + r_2k - r_3k - r_1j - r_2n \\
&= k\left(r_1 + r_2 - r_3 - \frac{e}{p-e}\right) - r_1j - r_2n \\
&\geq 2 - r_1j - r_2n.
\end{aligned}$$

This proves the Lemma. □

Corollary 3.2.34. *Let* $\max\left\{4, \frac{6e}{p-1}\right\} \leq 2r < r_t$ *and denote by* ν *the valuation* $\nu_{U_t^m(r,r_t)}$. *Consider the expression*

$$\gamma_n^m E^n \gamma_j^m F^j = \gamma_j^m F^j \gamma_n^m E^n + \sum_{\mu \neq (0,j,n,0)} a_\mu D_\mu^m.$$

Then $\nu\left(a_\mu D_\mu^m\right) > -r(j+n)$. *Moreover there exists a* $c > 0$ *independent of* n, j *such that*

$$\nu\left(a_\mu D_\mu^m\right) \geq -r(j+n) + c|\mu - (0,j,n,0)|.$$

Proof. Use Lemma 3.2.33 with $r_1 = r_2 = r_3 = r$ and then apply Lemma 3.2.32. □

Lemma 3.2.35. *We abbreviate for this Lemma* $\nu = \nu_{U_t^m(r,r_t)}$.
Let $\max\left\{4, \frac{6e}{p-1}\right\} \leq 2r < r_t$. *Consider the equality in* U_t

$$D_\mu^m D_\eta^m = \sum_\alpha a_\alpha D_\alpha^m.$$

Then there exists a $c > 0$ *independent of* μ *and* η *such that*

$$\nu\left(a_\alpha D_\alpha^m\right) \geq \nu\left(D_\mu^m\right) + \nu\left(D_\eta^m\right) + c\big||\mu+\eta| - |\alpha|\big|.$$

In particular $D_\mu^m D_\eta^m \in U_t^m(r, r_t)$ *and*

$$\nu\left(D_\mu^m D_\eta^m\right) \geq \nu\left(D_\mu^m\right) + \nu\left(D_\eta^m\right).$$

Proof. Combine the the Lemmas 3.2.32 and 3.2.34. ☐

Proposition 3.2.36. *Let* $\max\left\{4, \frac{6e}{p-1}\right\} \leq 2r < r_t$. *Then* $U_t^m(r, r_t)$ *is a subalgebra of* U_t *and the valuation* $\nu_{U_t^m(r,r_t)}$ *is submultiplicative on* $U_t^m(r, r_t)$.

Proof. We have to show that $\sum a_\mu D_\mu^m, \sum a_\eta D_\eta^m \in U_t^m(r, r_t)$ implies that

$$\sum a_\mu D_\mu^m \cdot \sum a_\eta D_\eta^m \in U_t^m(r, r_t).$$

In U_t there is a series $\sum a_\alpha D_\alpha^m$ such that

$$\sum a_\mu D_\mu^m \cdot \sum a_\eta D_\eta^m = \sum a_\alpha D_\alpha^m.$$

We know from Lemma 3.2.35 that there exists a $c > 0$ such that

$$\nu\left(a_\alpha D_\alpha^m\right) \geq \min\left\{\nu\left(a_\mu D_\mu^m\right) + \nu\left(a_\eta D_\eta^m\right) + c\big||\mu+\eta| - |\alpha|\big| : \mu, \eta \in \mathbb{N}_0^5\right\}. \tag{3.2.17}$$

But the set

$$\left\{\nu\left(a_\mu D_\mu^m\right) + \nu\left(a_\eta D_\eta^m\right) : \mu, \eta \in \mathbb{N}_0^5\right\}$$

is bounded from below by $\nu\left(\sum_\mu a_\mu D_\mu^m\right) + \nu\left(\sum_\eta a_\eta D_\eta^m\right)$. We also have that

$$\min\left\{\nu\left(a_\mu D_\mu^m\right) + \nu\left(a_\eta D_\eta^m\right) : |\mu| + |\eta| \geq k\right\} \longrightarrow \infty$$

as $k \to \infty$. Hence (3.2.17) implies that also $\nu\left(a_\alpha D_\alpha^m\right) \to \infty$ as $|\alpha| \to \infty$ and thus $\sum a_\alpha D_\alpha^m \in U_t^m(r, r_t)$ and

$$\nu\left(\sum a_\mu D_\mu^m \cdot \sum a_\eta D_\eta^m\right) \geq \nu\left(\sum a_\mu D_\mu^m\right) + \nu\left(\sum a_\eta D_\eta^m\right).$$

☐

Proposition 3.2.37. *The space* $U_t^b(r, r_t)$ *is a subalgebra of* U_t *and* $\nu_{U_t^b(r,r_t)}$ *is submultiplicative.*

Proof. Statements that are very similar to the statements in the Lemmas 3.2.33-3.2.35 are also valid for $U_t^b(r, r_t)$. Their proofs are similar and easier as in the case of $U_t^m(r, r_t)$. $\square$

3.3 The analytic quantum distribution algebra of certain subgroups

In the case $q = 1$ there are two approaches to show that the distribution algebra is a Frechet-Stein algebra. One by P. Schneider and J. Teitelbaum in their paper [ST03]. They use Mahler series to approximate locally analytic functions on $\mathbb{Z}_p$ and to describe the distribution algebra in terms of elements of the group, which is in our case $H = \mathrm{GL}(2, \mathcal{O})$. With the help of this description they define norms on the distribution algebra and use filtrations associated to these norms to show the Fréchet Stein property by passing to the graded algebras which turn out to be commutative.
In contrast M. Emerton in his paper [Eme11] uses open compact subgroups and investigates the analytic distributions on such subgroups and maps between them for distinct subgroups. To show the Fréchet Stein property he uses two filtrations which are not induced from valuations. We will have to use some ingredients from both approaches but will mostly go along the lines of [Eme11].

Recall that we denoted the quantum analytic functions on the subgroup $B(e, r)$ by $C_q^{an}(e, r)$. In this section we will first describe $C_q^{an}(e, r)'_b$ in terms of monomials of Lie algebra elements. We then show that if we work with the space of overconvergent quantum locally analytic functions, its dual space $D_q^{\dagger}(e, r)$ can be written as projective limit of Noetherian Banach algebras with right flat transition maps. Thus we establish that $D_q^{\dagger}(e, r)$ is a Frechet-Stein algebra.

For the rest of this section let $h \in \mathcal{O}$ with $\nu(h) > 4 + \frac{6e}{p-1}$. Let $q = e^h$ which implies $\nu(h) = \nu(1-q)$. Recall that $l_n^m = \left\lfloor \frac{n}{p^m} \right\rfloor$ and $\gamma_n^m = \frac{l_n^m!}{n!}$. We use the same symbol e for the identity matrix and for $\nu(p)$ but it will be clear from the context which one we mean. Let $\max\left\{4 + \frac{6e}{p-1}\right\} \leq 2r < \nu(h)$. Note that we do not require $r \in \nu(K)$.

Definition 3.3.1. Recall that for $\mu \in \mathbb{N}_0^4$ we defined

$$D_\mu := \frac{H_1^{\mu_1} F^{\mu_2} E^{\mu_3} H_2^{\mu_4}}{\mu_1! \mu_2! \mu_3! \mu_4!}; \qquad \bar{D}_\mu := \binom{H_1}{\mu_1} \frac{F^{\mu_2} E^{\mu_3}}{\mu_2! \mu_3!} \binom{H_2}{\mu_4}.$$

We also define the following K-linear subspaces of $\prod_{\mu \in \mathbb{N}_0^4} K$.

$$D_q^b(e,r) := \left\{ \sum_\mu a_\mu D_\mu : a_\mu \in K,\ \nu(a_\mu) - r|\mu| \to \infty \text{ for } |\mu| \to \infty \right\}$$

$$D_q^{an}(e,r) := \left\{ \sum_\mu a_\mu \bar{D}_\mu : a_\mu \in K,\ \nu(a_\mu) - r|\mu| \text{ is bounded from below} \right\}$$

$$D_q^\dagger(e,r) := \varprojlim_{R<r} D_q^b(e,R).$$

Definition 3.3.2. On $D_q^{an}(e,r)$ we define a valuation by

$$\nu_{D_q^{an}(e,r)} \left(\sum_\mu a_\mu \bar{D}_\mu \right) := \inf\{\nu(a_\mu) - r|\mu|\}.$$

On $D_q^b(e,r)$ we define a valuation by

$$\nu_{D_q^b(e,r)} \left(\sum_\mu a_\mu D_\mu \right) := \inf\{\nu(a_\mu) - r|\mu|\}.$$

We endow $D_q^\dagger(e,r)$ with the locally convex projective limit topology.

Definition 3.3.3. Recall that for $m \in \mathbb{N}$ and $\mu \in \mathbb{N}_0^4$ we defined

$$D_\mu^m := \gamma_{\mu_1}^m \gamma_{\mu_2}^m \gamma_{\mu_3}^m \gamma_{\mu_4}^m H_1^{\mu_1} F^{\mu_2} E^{\mu_3} H_2^{\mu_4}.$$

We define

$$D_q^m(e,r) := \left\{ \sum_\mu a_\mu D_\mu^m : a_\mu \in K,\ \nu(a_\mu) - r|\mu| \to \infty \text{ for } |\mu| \to \infty \right\}.$$

On $D_q^m(e,r)$ we define a valuation by

$$\nu_{D_q^m(e,r)} \left(\sum_\mu a_\mu D_\mu^m \right) := \min\left\{\nu(a_\mu) - r|\mu|\right\}.$$

Lemma 3.3.4. *Let* $\max\left\{4, \frac{6e}{p-1}\right\} \leq 2r < \nu(h)$. *Then the K-vector spaces $D_q^b(e,r)$ and $D_q^m(e,r)$ can be endowed with a continuous K-Banach algebra structure such that we get canonical algebra embeddings $U_q(\mathfrak{gl}_2, K) \subset D_q^b(e,r)$ and $U_q(\mathfrak{gl}_2, K) \subset D_q^m(e,r)$. Moreover the valuations $\nu_{D_q^b(e,r)}$ and $\nu_{D_q^m(e,r)}$ are submultiplicative.*

Proof. We can define surjective continuous maps of K-vector spaces

$$
\begin{aligned}
U_t^b(r,\nu(h)) &\to D_q^b(e,r) \\
U_t^m(r,\nu(h)) &\to D_q^m(e,r)
\end{aligned}
$$

by sending $t \mapsto h$. Let $\phi_h : U \to D$ be one of these morphisms. Using the definition of U, D and ν_U, ν_D we can see that $\nu_D(\phi_h(g)) \geq \nu_U(g)$ for all $g \in U$. Since ν_U is submultiplicative this implies that the kernel is a two sided ideal and thus there is a canonical algebra structure induced on D.
By the definition of ν_U and ν_D for every $f \in D$ there exists $\mathfrak{F} \in U$ such that $\phi_h(\mathfrak{F}) = f$ and

$$\nu_D(f) = \nu_U(\mathfrak{F}).$$

Combining with $\nu_D(\phi_h(g)) \geq \nu_U(g)$ for all $g \in U$ we obtain

$$\nu_D(f) = \sup\{\nu_U(g) : \phi_h(g) = f\}.$$

This implies that ϕ_h is strict and thus ν_D is submultiplicative with respect to the induced algebra structure.
Let $R := K\{X_1, ..., X_6\}$ be the noncommutative polynomial ring in $X_1, ..., X_6$. Define a map $R \to D$ by sending $X_1, ..., X_6$ to $e^{hH_1}, e^{hH_2}, e^{-hH_1}, e^{-hH_2}, F, E$. This map factors through an inclusion $U_q(\mathfrak{gl}_2, K) \subset D$. □

Lemma 3.3.5. *For $r \in \mathbb{Q}$, $\max\left\{4, \frac{6e}{p-1}\right\} \leq 2r < \nu(h)$ let*

$$D := \left\{\sum_\mu a_\mu \bar{D}_\mu : a_\mu \in K,\ \nu(a_\mu) - r|\mu| \to \infty\right\} \subseteq \prod_{\mu \in \mathbb{N}_0^4} K$$

with valuation

$$\nu_D\left(\sum_\mu a_\mu \bar{D}_\mu\right) = \inf\{\nu(a_\mu) - r|\mu|\}.$$

We then have that

$$D_q^b(e,r) = D.$$

as K-Banach spaces.

Proof. By [PSS14] Proposition 2.4.5 we know that for $s \in \mathbb{N}$, $s \geq 2$ we have that in $D_q^b(e,r)$

$$p^{sN}\binom{H_i}{N} = p^{sN}\frac{H_i}{N!} + \sum_{k=0}^{N-1} a_k \frac{(p^s H_1)^k}{k!}$$

for some $a_k \in \mathbb{Z}_p$. Their proof also works for π^s instead of p^s for $\nu(\pi^s) > \frac{e}{p-1}$. Since $r \in \mathbb{Q}$ we can enlarge the field and assume that $r \in \mathbb{Z}$. Then we can conclude that

$$\binom{H_i}{N} = \frac{H_i}{N!} + \sum_{k=0}^{N-1} a_k \frac{H_i^k}{k!} \tag{3.3.1}$$

with $\nu_{D_q^b(e,r)}\left(a_k \frac{H_i^k}{k!}\right) \geq -rN$. Since $\left\{\sum_{|\mu|\leq k} a_\mu \bar{D}_\mu : a_\mu \in K \text{ and } k \in \mathbb{N}_0\right\}$ is dense in both spaces we only have to show that the valuations on this set coincide. Note that here we already used that $D_q^b(e,r)$ is a K-algebra. Because of the estimates of the summands in equation (3.3.1) we can conclude that

$$\nu_{D_q^b(e,r)}(g) \geq \nu_D(g)$$

for all $g \in \left\{\sum_{|\mu|\leq k} a_\mu \bar{D}_\mu : a_\mu \in K \text{ and } k \in \mathbb{N}_0\right\}$.
Let $f = \sum_{|\mu|\leq k} a_\mu \bar{D}_\mu \in \left\{\sum_{|\mu|\leq k} a_\mu \bar{D}_\mu : a_\mu \in K \text{ and } k \in \mathbb{N}_0\right\}$. Let

$$l = \max\left\{i \in \mathbb{N}_0 : \exists \mu \in \mathbb{N}_0^4 \text{ with } i = |\mu| \text{ and } \nu_D\left(a_\mu \bar{D}_\mu\right) = \nu_D(f)\right\}$$

We have that

$$\sum_{|\mu|=l} a_\mu \bar{D}_\mu = \sum_{|\mu|=l} a_\mu D_\mu + \sum_{|\mu|<l} b_\mu D_\mu$$

for some b_μ. Here we have that $\nu_{D_q^b(e,r)}\left(\sum_{|\mu|=l} a_\mu D_\mu\right) = \nu_D(f)$ by (3.3.1) and $\nu_{D_q^b(e,r)}\left(\sum_{|\mu|<l} b_\mu D_\mu\right) \geq \nu_D(f)$. Moreover $\sum_{|\mu|<l} a_\mu \bar{D}_\mu = \sum_{|\mu|<l} c_\mu D_\mu$ for some $c_\mu \in K$ with $\nu_{D_q^b(e,r)}\left(\sum_{|\mu|<l} c_\mu D_\mu\right) \geq \nu_D(f)$. But this means that

$$\begin{aligned}\nu_{D_q^b(e,r)}\left(\sum_{|\mu|\leq l} a_\mu \bar{D}_\mu\right) &= \nu_{D_q^b(e,r)}\left(\sum_{|\mu|<l} b_\mu D_\mu + \sum_{|\mu|<l} c_\mu D_\mu + \sum_{|\mu|=l} a_\mu D_\mu\right) \\ &= \nu_D(f).\end{aligned}$$

Since also

$$\nu_{D_q^b(e,r)}\left(\sum_{l<|\mu|\leq k} a_\mu \bar{D}_\mu\right) \geq \nu_D\left(\sum_{l<|\mu|\leq k} a_\mu \bar{D}_\mu\right) > \nu_D(f)$$

we can conclude that

$$\nu_{D_q^b(e,r)}(f) = \nu_{D_q^b(e,r)}\left(\sum_{|\mu|\leq l} a_\mu \bar{D}_\mu + \sum_{l<|\mu|\leq k} a_\mu \bar{D}_\mu\right) = \nu_D(f)$$

□

Remark 3.3.6. We recall some notation concerning the quantum analytic functions around the identity matrix. Let $e^\mu := (a-1)^{\mu_1} b^{\mu_2} c^{\mu_3} (d-1)^{\mu_4}$ for $\mu \in \mathbb{N}_0^4$.

For $r \in \mathbb{Q}$, $2 \leq 2r < \nu(h)$ and $2 \leq 2r' \leq \nu(h)$ we define

$$C_q^{an}(e,r) := \left\{ \sum_{\mu} a_\mu e^\mu : \nu(a_\mu) + r|\mu| \to \infty \right\}$$
$$C_q^{\dagger}(e,r') := \varinjlim_{r<r'} C_q^{an}(1,r).$$

3.3.7. Now we can describe the strategy in order to prove that $C_q^{\dagger}(e,r')'_b$ is a Fréchet Stein algebra in more detail. In Proposition 3.3.10 we will show that $C_q^{an}(e,r)'_b$ is isomorphic to $D_q^{an}(e,r)$ as K-Banach space.
Since we know by Lemma 2.1.11 that for $2r_1 \leq 2r_2 < \nu(h)$ the algebra morphism $C_q^{an}(g,r_1) \longrightarrow C^{an}(g,r_2)$ is continuous, injective and compact, [Sch02] Proposition 16.10 (see Proposition 1.1.13) implies that we have equalities of locally convex K-vector spaces

$$C_q^{\dagger}(e,r')'_b = \varprojlim_{r<r'} C_q^{an}(1,r)'_b = \varprojlim_{r<r'} D_q^{an}(e,r).$$

As we will see in **3.3.12** we can find a sequence $(r_n, m_n)_{n\in\mathbb{N}} \in (\mathbb{Q}\times\mathbb{N})^{\mathbb{N}}$ such that $\left\{D_q^{an}(e,r) : r < r'\right\}$ and $\left\{D_q^{m_n}(e,r_n) : n \in \mathbb{N}\right\}$ are cofinal systems. Hence we can conclude that

$$C_q^{\dagger}(e,r')'_b = \varprojlim_{n} D_q^{m_n}(e,r_n).$$

Since $C_q^{an}(e,r)$ is a Banach Hopf algebra we have a K-algebra structure on $C_q^{an}(e,r)'_b$. We have inclusions

$$D_q^{m_n}(e,r) \subseteq D_q^{b}(e,r) \subseteq D_q^{an}(e,r) = C_q^{an}(e,r)'_b$$

and we will see in **3.3.9** that $D_q^{m_n}(e,r) \subseteq C_q^{an}(e,r)'_b$ is an inclusion of K-algebras. Thus we can conclude that $C_q^{\dagger}(e,r')'_b = \varprojlim_n D_q^{m_n}(e,r_n)$ also as K-algebras.
In Proposition 3.3.16 we will show that $D_q^{m_n}(e,r_n)$ is Noetherian and in Proposition 3.3.19 we will see that $D_q^{m_{n+1}}(e,r_{n+1}) \to D_q^{m_n}(e,r_n)$ is right flat. Hence we can conclude that $C_q^{\dagger}(e,r')'_b$ is a Fréchet Stein algebra.

3.3.8. Let $\max\left\{4, \frac{6e}{p-1}\right\} \leq 2r < \nu(h)$. Recall that for $r_t \geq \frac{2e}{p-1}$ we defined the $K[[t]]_{r_t}$-module $C_t^{an}(e,r,r_t)$ at the end of chapter 2. The continuous $K[[t]]$-linear bracket

$$\langle\cdot,\cdot\rangle : U_t \times M_q(2,t) \longrightarrow K[[t]]$$

restricts to a continuous $K[[t]]_{\nu(h)}$-linear bracket

$$\langle\cdot,\cdot\rangle : U_t^{an}(r,\nu(h)) \times C_t^{an}(e,r,\nu(h)) \longrightarrow K[[t]]$$

and the image of the latter map is contained in $K[[t]]_{\nu(h)}$ by Theorem 3.2.25. There exists a unique continuous K-algebra morphism $K[[t]]_{\nu(h)} \to K$ sending t to h. Let $\langle \cdot, \cdot \rangle_q$ be the unique map such that

$$\begin{tikzcd}
U_t^{an}(r,\nu(h)) \times C_t^{an}(e,r,\nu(h)) \arrow[r, "\langle \cdot,\cdot \rangle"] \arrow[d] & K[[t]]_{\nu(h)} \arrow[d] \\
D_q^{an}(e,r) \times C_q^{an}(e,r) \arrow[r, "\langle \cdot,\cdot \rangle_q"] & K
\end{tikzcd}$$

commutes. The map $\langle \cdot, \cdot \rangle_q$ is a continuous K-vector space bracket. If we restrict it to $U_q(\mathfrak{gl}_2) \times M_q(2,K)$ we recover the usual pairing considered e.g. in [KS97] chapter 9.4 Theorem 18. By abuse of notation we will write $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_q$.

3.3.9. The bracket $D_q^{an}(e,r) \times C_q^{an}(e,r) \to K$ restricts to a bracket

$$D_q^{m}(e,r) \times C_q^{an}(e,r) \longrightarrow K.$$

This bracket induces a map $D_q^m(e,r) \longrightarrow C_q^{an}(e,r)'_b$. Since $C_q^{an}(e,r)$ is a bialgebra there is a natural algebra structure on $C_q^{an}(e,r)'_b$ given by

$$\lambda\mu(f) = \lambda \otimes \mu(\Delta(f)); \qquad\qquad 1 = \epsilon.$$

Since the bracket $\langle \cdot, \cdot \rangle : U_t \times M_q(2,t) \longrightarrow K[[t]]$ fulfills (3.2.1) we know that for $\mu, \eta, \delta \in \mathbb{N}_0^4$ we have that

$$\left\langle D_\mu^m D_\eta^m, e^\delta \right\rangle = \left\langle D_\mu^m \otimes D_\eta^m, \Delta\left(e^\delta\right)\right\rangle = \langle D_\mu^m, \cdot \rangle \otimes \langle D_\eta^m, \cdot \rangle \left(\Delta\left(e^\delta\right)\right)$$

and $\langle 1, e^\delta \rangle = \epsilon\left(e^\delta\right)$. We know that the K-linear span of $\left\{D_\mu^m : \mu \in \mathbb{N}_0^4\right\}$ is dense in $D_q^m(e,r)$ and that the K-linear span of $\left\{e^\delta : \delta \in \mathbb{N}_0^4\right\}$ is dense in $C_q^{an}(e,r)$. Thus the map $D_q^m(e,r) \longrightarrow C_q^{an}(e,r)'_b$ is a K-algebra morphism.

Proposition 3.3.10. *Let* $\max\left\{4, \frac{6e}{p-1}\right\} \le 2r < \nu(h)$. *Via the bracket we can identify* $C_q^{an}(e,r)'_b$ *with* $D_q^{an}(e,r)$ *as* K*-Banach spaces.*

Proof. By Lemma 1.2.3 it suffices to show that

1. $\nu\left(\left\langle \pi^{\lfloor r|\mu| \rfloor}\bar{D}_\mu, \pi^{-\lfloor r|\mu|\rfloor} e^\mu \right\rangle\right) = 0$ for all $\mu \in \mathbb{N}_0^4$
2. There exists $\epsilon > 0$ such that for all $\eta \neq \mu$

$$\nu\left(\left\langle \pi^{\lfloor r|\eta| \rfloor}\bar{D}_\eta, \pi^{-\lfloor r|\mu|\rfloor} e^\mu \right\rangle\right) \ge \max\left\{1, \epsilon|\eta - \mu|\right\}.$$

By Lemma 3.2.24 there exists an $a \in \mathcal{O}$ such that

$$\left\langle \bar{D}_\mu, e^\mu \right\rangle = \frac{1}{\mu_2!\mu_3!}\frac{(q^2,q^2)_{\mu_2}(q^2,q^2)_{\mu_3}}{(1-q^2)^{\mu_2+\mu_3}}(1+ha).$$

Since by Lemma 3.1.4 we have that $\nu(n!) = \nu\left(\frac{(q^2,q^2)_n}{(1-q^2)^n}\right)$ and since we have that $\nu(1+ha) = 0$ we can conclude that $\nu\left(\langle \pi^{\lfloor r|\mu|\rfloor}\bar{D}_\mu, \pi^{-\lfloor r|\mu|\rfloor}e^\mu\rangle\right) = 0$.
2. follows directly from Theorem 3.2.25. $\square$

3.3.11. We will view $D_q^{an}(e,r)$ as K-Banach algebra with K-algebra structure induced by the equality $D_q^{an}(e,r) = C_q^{an}(e,r)'_b$. The latter carries a K-Banach algebra structure since $C_q^{an}(e,r)$ is a K-Banach Hopf algebra with valuation increasing comultiplication.

3.3.12. We will now describe a sequence $(r_n, m_n)_{n\in\mathbb{N}} \subset (\mathbb{Q}\times\mathbb{N})^{\mathbb{N}}$ such that $D_q^\dagger(e,r) = \varprojlim_n D_q^{m_n}(e,r_n)$.
First note that by Lemma 3.3.5 we know that $D_q^b(e,R) \subseteq D_q^{an}(e,R)$ and that for $R' > R$ we have that $D_q^{an}(e,R') \subseteq D_q^b(e,R)$ and that both inclusions are valuation increasing inclusions.
Thus for $\max\left\{4, \frac{6e}{p-1}\right\} \le 2r \le \nu(h)$ we have cofinal systems

$$\left\{D_q^b(e,R) : \max\left\{4, \frac{6e}{p-1}\right\} < 2R < 2r \le \nu(h)\right\}$$

and

$$\left\{D_q^{an}(e,R) : \max\left\{4, \frac{6e}{p-1}\right\} < 2R < 2r \le \nu(h)\right\}$$

and thus

$$D_q^\dagger(e,r) = \varprojlim_{R<r} D_q^{an}(e,R) = \varprojlim_{R<r} D_q^b(e,R)$$

as locally convex K-vector spaces.
Let $\max\left\{4, \frac{6e}{p-1}\right\} \le 2r_1 < 2r_2 < \nu(h)$. Let $m \in \mathbb{N}$ with $r_1 < r_2 - \frac{1}{p^m}$. Then the inequality $0 \le \nu(l_n^m!) \le \frac{\left\lfloor \frac{n}{p^m}\right\rfloor}{p-1} \le \frac{n}{p^m}$ implies

$$\nu_{D_q^m(e,r_1)}\left(D_\mu^m\right) = -r_1|\mu| > \left(\frac{1}{p^m} - r_2\right)|\mu| \ge \sum_{i=1}^4 \nu\left(l_{\mu_i}^m!\right) - r_2|\mu| = \nu_{D_q^b(e,r_2)}\left(D_\mu^m\right)$$

and thus we have continuous inclusions

$$D_q^b(e,r_2) \subseteq D_q^m(e,r_1) \subseteq D_q^b(e,r_1).$$

This implies that for $\max\left\{4, \frac{6e}{p-1}\right\} \le 2r_1 < 2r_2 < 2r_3 < \nu(h)$ there are $m_2 \ge m_1$ such that there are continuous inclusions

$$D_q^b(e,r_3) \subseteq D_q^{m_2}(e,r_2) \subseteq D_q^b(e,r_2) \subseteq D_q^{m_1}(e,r_1) \subseteq D_q^b(e,r_1).$$

Fix an increasing sequence $\{r_n\}_{n\in\mathbb{N}}$ with $r_n \in \mathbb{Z}\frac{1}{p^{j_n}}$ for some $j_n \in \mathbb{N}$ such that $\frac{6e}{p-1} \le 2r_n < 2r \le \nu(h)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} r_n = r$. Using the statement above we can find an increasing sequence $\{m_n\}_{n\in\mathbb{N}}$ such that $m_n \geq j_n$ and

$$D_q^b(e, r_{n+2}) \subseteq D_q^{m_{n+1}}(e, r_{n+1}) \subseteq D_q^b(e, r_{n+1}) \subseteq D_q^{m_n}(e, r_n) \subseteq D_q^b(e, r_n).$$

Then $\left\{D_q^{m_n}(e, r_n) : n \in \mathbb{N}\right\}$ and $\left\{D_q^b(e, R) : \max\left\{4, \frac{6e}{p-1}\right\}\right\}$ are cofinal systems. Thus also $\left\{D_q^{m_n}(e, r_n) : n \in \mathbb{N}\right\}$ and

$$\left\{D_q^{an}(e, R) : \max\left\{4, \frac{6e}{p-1}\right\} < 2R < 2r \leq \nu(h)\right\}$$

are cofinal systems. This implies that

$$D_q^{\dagger}(e, r) = \varprojlim_{R<r} D_q^{an}(e, R) = \varprojlim_{n} D_q^{m_n}(e, r_n).$$

Since we know by **3.3.9** that $D_q^{m_n}(e, r_n) \subseteq C_q^{an}(e, R)_b'$ is an inclusion of K-Banach algebras for $r_n \leq R < r$, we can conclude that $D_q^{\dagger}(e, r) = \varprojlim_n D_q^{m_n}(e, r_n)$ is an equality of locally convex K-algebras.
Hence in order to show that $D_q^{\dagger}(e, r)$ is a Fréchet Stein algebra we only have to show that $D_q^{m_n}(e, r_n)$ is Noetherian and that

$$D_q^{m_{n+1}}(e, r_{n+1}) \longrightarrow D_q^{m_n}(e, r_n)$$

is right flat. We will show this by using filtration techniques.

Definition 3.3.13. Let R be a unital ring. A filtration on R is a family $(\mathrm{F}^s R)_{s\in\mathbb{R}}$ of additive subgroups $\mathrm{F}^s R \subseteq R$ such that for all $r, s \in \mathbb{R}$,

1. $\mathrm{F}^r R \subseteq \mathrm{F}^s R$ if $r \geq s$,
2. $\mathrm{F}^r R \cdot \mathrm{F}^s R \subseteq \mathrm{F}^{r+s} R$,
3. $\cup_{s\in\mathbb{R}} \mathrm{F}^s R = R$ and $1 \in \mathrm{F}^0 R$.

For $s \in \mathbb{R}$ we define $\mathrm{gr}^s R = \mathrm{F}^s R / \cup_{s'>s} \mathrm{F}^{s'} R$ and $\mathrm{gr}^\bullet R = \oplus_{s\in\mathbb{R}} \mathrm{gr}^s R$. R is called complete if $R \cong \varprojlim_s R/\mathrm{F}^s R$. The filtration $\mathrm{F}^\bullet$ is called quasi integral if there exists an $n_0 \in \mathbb{N}$ such that

$$\{s \in \mathbb{R} : \mathrm{gr}^s R \neq 0\} \subseteq \mathbb{Z} \cdot \frac{1}{n_0}.$$

Proposition 3.3.14 ([ST03] Proposition 1.1)**.** *Let R be a complete filtered ring whose filtration is quasi integral. If $\mathrm{gr}^\bullet R$ is (left) Noetherian so is R.*

We will also need the following Lemma.

Lemma 3.3.15 ([Eme11] Lemma 5.2.2)**.** *Let $n = p^m l_n^m + r$ with $0 \leq r < p^m$. Then the valuation of $n!$ can be computed as $\nu(n!) = l_n^m \nu(p^m!) + \nu(r!) + \nu(l_n^m!)$.*

In particular for an indeterminate X we have

$$\left(\frac{X^{p^m}}{p^m!}\right)^{l_n^m}\frac{X^r}{r!} = a\frac{l_n^m!}{n!}X^n$$

for some $a \in \mathbb{Q}$ with $\nu(a) = 0$.

Proposition 3.3.16. *Let $m, n \in \mathbb{N}$ with $m \geq n$ and let $r \in \mathbb{Z} \cdot \frac{1}{p^n}$ be such that $\max\left\{4, \frac{6e}{p-1}\right\} \leq 2r < \nu(h)$. Let $A^m := \{f \in D_q^m(e,r) : \nu_{D_q^m(e,r)}(f) \geq 0\}$. Then A^m and $D_q^m(e,r)$ are Noetherian.*

Proof. We will abbreviate $\nu_{D_q^m(e,r)}$ by $\nu_{m,r}$ in this proof. Let $\mathrm{F}^\bullet$ be the valuation filtration on A^m i.e.

$$\mathrm{F}^b A^m = \{f \in A^m : \nu_{m,r}(f) \geq b\}$$

for $b \geq 0$ and $\mathrm{F}^b A^m = A^m$ if $b \leq 0$. Since $r \in \mathbb{Z} \cdot \frac{1}{p^n}$ this filtration is quasi integral and with it A^m is a complete filtered ring. Thus by Proposition 3.3.14 it suffices to show that $\mathrm{gr}^\bullet A^m$ is Noetherian. Therefore it suffices to show that $\mathrm{gr}^\bullet A^m$ is a finitely generated commutative $\kappa := \mathcal{O}/\pi$ algebra.
Let

$$M := \left\{\gamma_l^m H_1^l, \gamma_l^m H_2^l, \gamma_l^m \mathrm{F}^l, \gamma_l^m E^l : \quad l \in \mathbb{N}\right\}.$$

The Lemmas 3.2.33 and 3.2.32 imply that $\nu_{m,r}(XY) = \nu_{m,r}(YX)$ and

$$\nu_{m,r}([X,Y]) > \nu_{m,r}(X) + \nu_{m,r}(Y)$$

for $X, Y \in M$. Consider a finite product $X_1 \cdots X_n$ for $X_i \in M$. Assume that $\nu_{m,r}(X_1 \cdots X_n) = \sum_{i=1}^n \nu_{m,r}(X_i)$. Then $\nu_{m,r}([X,Y]) > \nu_{m,r}(X) + \nu_{m,r}(Y)$ for $X, Y \in M$ and the submultiplicativity of $\nu_{m,r}$ imply that

$$\nu_{m,r}\left(X_1 \cdots X_n - X_{\sigma(1)} \cdots X_{\sigma(n)}\right) > \nu_{m,r}(X_1 \cdots X_n) \tag{3.3.2}$$

for all $\sigma \in S_n$. This also implies that if $\nu_{m,r}\left(X_{\sigma(1)} \cdots X_{\sigma(n)}\right) > \sum_{i=1}^n \nu_{m,r}(X_i)$ for some $\sigma \in S_n$, then $\nu_{m,r}\left(X_{\tau(1)} \cdots X_{\tau(n)}\right) > \sum_{i=1}^n \nu_{m,r}(X_i)$ for every $\tau \in S_n$.
Denote by

$$\psi : A^m \longrightarrow \mathrm{gr}^\bullet A^m$$

the principal symbol map sending $f \in \mathrm{F}^b A^m \setminus \cup_{b'>b} \mathrm{F}^{b'} A^m$ to its residue class in $\mathrm{F}^b A^m / \cup_{b'>b} \mathrm{F}^{b'} A^m$.
Let $\mu, \eta \in \mathbb{N}_0^4$, $a_\mu, a_\eta \in K$ and assume $\psi\left(a_\mu D_\mu^m\right)\psi\left(a_\eta D_\eta^m\right) = 0$ in $\mathrm{gr}^\bullet A^m$. Then the considerations above imply that also $\psi\left(a_\eta D_\eta^m\right)\psi\left(a_\mu D_\mu^m\right) = 0$.
If $\psi\left(a_\mu D_\mu^m\right)\psi\left(a_\eta D_\eta^m\right) \neq 0$ in $\mathrm{gr}^\bullet A^m$ then they imply that

$$\psi\left(a_\mu D_\mu^m\right)\psi\left(a_\eta D_\eta^m\right) = \psi\left(a_\eta D_\eta^m\right)\psi\left(a_\mu D_\mu^m\right).$$

Thus $\operatorname{gr}^\bullet A^m$ is commutative.
Now we want to show that $\operatorname{gr}^\bullet A^m$ is finitely generated over $\kappa = \mathcal{O}/\pi$. We claim that the images under ψ of the elements in

$$N := \left\{ \pi^{\lceil r|\mu| \rceil} \frac{H_1^{\mu_1} F^{\mu_2} E^{\mu_3} H_2^{\mu_4}}{\mu_1!\mu_2!\mu_3!\mu_4!} : \quad \mu_i \leq p^m \right\}$$

together with $\psi(\pi)$ generate $\operatorname{gr}^\bullet A^m$ as a κ-algebra. For $\mu \in \mathbb{N}_0^4$ define

$$B_\mu := \pi^{\lceil r|\mu| \rceil} \gamma^m_{\mu_1} \gamma^m_{\mu_2} \gamma^m_{\mu_3} \gamma^m_{\mu_4} H_1^{\mu_1} F^{\mu_2} E^{\mu_3} H_2^{\mu_4}$$

Since the elements of the form $\psi(\pi)^s \psi(B_\mu)$ for $s \in \mathbb{N}_0$ generate $\operatorname{gr}^\bullet A^m$ as κ-vector space, it is enough to show that in $\operatorname{gr}^\bullet A^m$ every B_μ can be expressed as a product of elements in $\psi(N)$.
Recall that $\mu_i = p^m l^m_{\mu_i} + r_{\mu_i}$ and that $r \in \mathbb{Z}\frac{1}{p^n}$ for $m \geq n$. Thus

$$\lceil r|\mu| \rceil = \sum_i r p^m l^m_{\mu_i} + \left\lceil r \sum_i r_{\mu_i} \right\rceil \tag{3.3.3}$$

with $rp^m \in \mathbb{N}_0$.
We will suppress the map ψ in the next equation to make it more readable. By using equation (3.3.2), equation (3.3.3), the commutativity of $\operatorname{gr}^\bullet A^m$ and Lemma 3.3.15 we obtain that there exists $a \in \mathcal{O}^\times$ such that in $\operatorname{gr}^\bullet A^m$

$$\begin{aligned}
B_\mu &= \pi^{\lceil r|\mu| \rceil} \gamma^m_{\mu_1} \gamma^m_{\mu_2} \gamma^m_{\mu_3} \gamma^m_{\mu_4} H_1^{\mu_1} F^{\mu_2} E^{\mu_3} H_2^{\mu_4} \\
&= \pi^{\lceil r|\mu| \rceil} \frac{l^m_{\mu_1}! l^m_{\mu_2}! l^m_{\mu_3}! l^m_{\mu_4}!}{\mu_1!\mu_2!\mu_3!\mu_4!} H_1^{\mu_1} F^{\mu_2} E^{\mu_3} H_2^{\mu_4} \\
&= a \left(\frac{\pi^{rp^m} H_1^{p^m}}{p^m!} \right)^{l^m_{\mu_1}} \left(\frac{\pi^{rp^m} F^{p^m}}{p^m!} \right)^{l^m_{\mu_2}} \left(\frac{\pi^{rp^m} E^{p^m}}{p^m!} \right)^{l^m_{\mu_3}} \left(\frac{\pi^{rp^m} H_2^{p^m}}{p^m!} \right)^{l^m_{\mu_4}} \\
&\cdot \pi^{\lceil r \sum_i r_{\mu_i} \rceil} \frac{H_1^{r_{\mu_1}}}{r_{\mu_1}!} \frac{F^{r_{\mu_2}}}{r_{\mu_2}!} \frac{E^{r_{\mu_3}}}{r_{\mu_3}!} \frac{H_2^{r_{\mu_4}}}{r_{\mu_4}!}.
\end{aligned}$$

We see that the right hand side can be written as product with factors in N. Thus $\operatorname{gr}^\bullet A^m$ is a finitely generated κ-algebra and hence Noetherian. Using Proposition 3.3.14 we can conclude that A^m is Noetherian. In the same way one can show that $D^m_q(e, r)$ is Noetherian. But in this case one can simplify the proof by extending the field in such a way that one can assume that $r \in \mathbb{Z}$. □

3.3.17. In order to prove flatness will consider the following kind of filtration. Let A be a possibly noncommutative $\mathbb{Z}_p$-algebra. Let B be an A-algebra with an exhaustive increasing filtration $F_0 \subseteq F_1 \subseteq F_2 \cdots$ of $\mathbb{Z}_p$-submodules. Consider the following conditions:

1. $F_i F_j \subseteq F_{i+j}$ for all $i, j \geq 0$
2. $F_0 = A$
3. The associated graded algebra $\mathrm{gr}^\bullet B$ is finitely generated over A by central elements.

Proposition 3.3.18 ([Eme11] Proposition 5.3.10.)**.** *Let A be a p-torsion free p-adically separated left Noetherian $\mathbb{Z}_p$-algebra and let B be a $\mathbb{Z}_p$-subalgebra of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A$ containing A. Denote by $\hat{A}$ resp. $\hat{B}$ the p-adic completions of A resp. B. Assume B admits a filtration as in **3.3.17**. Then*

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A} \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{B}$$

is right flat.

Proposition 3.3.19. *Let $r_1, r_2 \in \mathbb{Z}_{\frac{1}{p^n}}$ for some $n \in \mathbb{N}_{\geq 3}$ with $r_2 \leq r_1 + \frac{e}{p-1}$ and $4 + \frac{6e}{p-1} \leq 2r_1 < 2r_2 < \nu(h)$. Let $m_1, m_2 \in \mathbb{N}$ with $m_2 > m_1 \geq n$ be such that $D_q^{m_2}(e, r_2) \subseteq D_q^{m_1}(e, r_1)$. Then the inclusion map*

$$D_q^{m_2}(e, r_2) \to D_q^{m_1}(e, r_1)$$

is right flat.

Proof. We will use the previous Proposition. We denote the valuation on $D_q^{m_i}(e, r_i)$ by ν_i. We define $A := \{f \in D_q^{m_2}(e, r_2) : \nu_2(f) \geq 0\}$ and thus $\hat{A} = A$. Then

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A = K \otimes_{\mathcal{O}} A = D_q^{m_2}(e, r_2) \subset D_q^{m_1}(e, r_1)$$

and hence we can evaluate ν_1 on elements of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A$. We define

$$B := \{f \in \mathbb{Q}_p \otimes_{\mathbb{Z}_p} A : \nu_1(f) \geq 0\}.$$

Then $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{B} = K \otimes_{\mathcal{O}} \hat{B} = D_q^{m_1}(e, r_1)$ and

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{A} \longrightarrow \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{B}$$

equals

$$D_q^{m_2}(e, r_2) \longrightarrow D_q^{m_1}(e, r_1).$$

So we are left to check the conditions of Proposition 3.3.18. Obviously A is p-adically separated and p-torsion free. That it is Noetherian is the content of Proposition 3.3.16 and $\nu_1(f) \geq \nu_2(f)$ for all $f \in D_q^{m_2}(e, r_2)$ implies that $B \supseteq A$.
Thus it suffices to construct a filtration as in **3.3.17**. Our filtration will be similar to the one used in [Eme11] Proposition 5.3.11. But note that we can't

work with an algebra generated by elements of the Lie algebra since some commutators are infinite series in our case.
For an element

$$f = \sum_{\mu} a_\mu D_\mu^{m_2} \in D_q^{m_2}(e, r_2)$$

let

$$\deg(f) := \sup \left\{ a \in \mathbb{N}_0 : \exists \mu \text{ such that } |\mu| = a \text{ and } a_\mu \neq 0 \right\} \in \mathbb{N}_0 \cup \{\infty\}.$$

We define for $l \in \mathbb{N}_0$

$$B_l := \{ f \in B : \deg(f) \leq l \}.$$

We define a filtration $\mathrm{F}_\bullet$ on B by setting

$$\mathrm{F}_l := A B_l.$$

We want to show that $\mathrm{F}_\bullet$ is as in **3.3.17**. We first show that $F_\bullet$ is exhaustive. Therefore let $f = \sum_\mu a_\mu D_\mu^{m_2} \in B \subseteq D_q^{m_2}(e, r_2)$. Since the sum converges in $D_q^{m_2}(e, r_2)$ there exists a $k \in \mathbb{N}$ such that $\sum_{|\mu| \geq k} a_\mu D_\mu^{m_2} \in A$ and thus

$$f = \sum_{|\mu| < k} a_\mu D_\mu^{m_2} + \sum_{|\mu| \geq k} a_\mu D_\mu^{m_2} \in A B_k = F_k$$

and hence we showed that $\mathrm{F}_\bullet$ is exhaustive. Since $\mathrm{F}_0 = A B_0 = A$ also property 2 of **3.3.17** is true.

We will show property 1 in two steps.
Step 1: We show that $A B_l = B_l A$.
We only show $A B_l \subseteq B_l A$ since the other inclusion is proven in exactly the same way. In order to prove Step 1 we will introduce some notation that is also useful for Step 2. We will denote by $X^{i,j}$ the expression $\gamma_i^{m_j} X^i$ for $X \in \{H_1, F, E, H_2\}$ and $j \in \{1, 2\}$. As an auxiliary result we will first show the following claim.
Claim 1: For $X, Y \in \{H_1, F, E, H_2\}$, $i \in \mathbb{N}_0$ and $l \in \mathbb{N}_0$ there exists $a_k \in K$ and elements $g_k \in D_q^{m_2}(e, r_2)$ for $0 \leq k \leq i$ such that

$$X^{l,2} Y^{i,1} = \sum_{k \leq i} a_k Y^{k,1} g_k$$

and $\nu_1 \left(a_k Y^{k,1} \right) \geq \nu_1 \left(Y^{i,1} \right)$ and $\nu_2 \left(g_k \right) \geq \nu_2 \left(X^{l,2} \right)$.
For $X, Y \in \{H_1, H_2\}$ the claim is immediate since H_1 and H_2 commute. Let

$Y \in \{H_1, H_2\}$ and $X = F$. Then we have

$$\begin{aligned}
F^{l,2} H_s^{i,1} &= F^{l,2} \gamma_i^{m_1} H_s^i \\
&= \gamma_i^{m_1} (H_s + (-1)^s l)^i F^{l,2} \\
&= \gamma_i^{m_1} \sum_{k=0}^{i} \binom{i}{k} (-1)^{s(i-k)} l^{i-k} H_s^i F^{l,2} \\
&= \sum_{k=0}^{i} \frac{l_i^{m_1}!}{k!(i-k)!} (-1)^{s(i-k)} l^{i-k} H_s^i F^{l,2} \\
&= \sum_{k=0}^{i} \frac{l_i^{m_1}!}{l_k^{m_1}!(i-k)!} (-1)^{s(i-k)} l^{i-k} H_s^{k,1} F^{l,2}. \qquad (3.3.4)
\end{aligned}$$

Using $\nu\left(l_i^{m_1}!\right) \geq \nu\left(l_k^{m_1}!\right)$ and $\nu(z!) = \frac{e(z-S(z))}{p-1}$ and $r_1 \geq \frac{3e}{p-1}$ we can conclude that

$$\nu_1\left(\frac{l_i^{m_1}!}{l_k^{m_1}!(i-k)!}(-1)^{s(i-k)} l^{i-k} H_s^{k,1}\right) \geq -\frac{e(i-k)}{p-1} - r_1 k \geq -r_1 i = \nu_1\left(H_s^{i,1}\right)$$

and thus Claim 1 follows in this case. The same strategy also works in the case $X = E$ and $Y \in \{H_1, H_2\}$ and in the case $X \in \{H_1, H_2\}$ and $Y \in \{E, F\}$. Now let $Y = F$ and let $X = E$. By the second statement of Lemma 3.2.33 we know that

$$\begin{aligned}
E^{l,2} F^{i,1} = &\sum_{k=1}^{\min\{l,i\}} \sum_{\sigma,\tau} a_{k,\tau,\sigma} F^{i-k,1} E^{l-k,2} H_1^{\tau,2} H_2^{\sigma,2} \\
&+ F^{i,1} E^{l,2} \qquad (3.3.5)
\end{aligned}$$

for some $a_{k,\tau,\sigma}$ with

$$\nu\left(a_{k,\tau,\sigma}\right) - r_1(i-k) - r_2(l-k+\tau+\sigma) \geq -r_1 i - r_2 l + 2.$$

This implies

$$\nu\left(a_{k,\tau,\sigma}\right) + \lfloor r_1 k \rfloor - \lceil r_2(-k+\tau+\sigma) \rceil \geq 0$$

and thus $b_{k,\tau,\sigma} := a_{k,\tau,\sigma} \pi^{\lfloor r_1 k \rfloor} \pi^{-\lceil r_2(-k+\tau+\sigma)\rceil} \in \mathcal{O}$. We then have that

$$a_{k,\tau,\sigma} F^{i-k,1} E^{l-k,2} H_1^{\tau,2} H_2^{\sigma,2} = b_{k,\tau,\sigma} \left[\frac{F^{i-k,1}}{\pi^{\lfloor r_1 k \rfloor}}\right] \left[\pi^{\lceil r_2(-k+\tau+\sigma)\rceil} E^{l-k,2} H_1^{\tau,2} H_2^{\sigma,2}\right].$$

Since

$$\nu_1\left(\frac{F^{i-k,1}}{\pi^{\lfloor r_1 k \rfloor}}\right) = -r_1(i-k) - \lfloor r_1 k \rfloor \geq -r_1 i = \nu_1\left(F^{i,1}\right)$$

and

$$\nu_2\left(\pi^{\lceil r_2(-k+\tau+\sigma)\rceil}E^{l-k,2}H_1^{\tau,2}H_2^{\sigma,2}\right)$$
$$\geq -r_2(l-k+\tau+\sigma)+\lceil r_2(-k+\tau+\sigma)\rceil \geq -r_2 l=\nu_2\left(E^{l,2}\right)$$

the claim is proven in this case. The proof of Claim 1 in the case $Y=E$ and $X=F$ is completely analogous and hence Claim 1 is proven.
Since we know that ν_1 and ν_2 are submultiplicative Claim 1 implies that for $X\in\{H_1,F,E,H_2\}$, $i\in\mathbb{N}_0$ and $\mu\in\mathbb{N}_0^4$ there exist $a_k\in K$ and $g_k\in D_q^{m_2}(e,r_2)$ for $0\leq k\leq i$ such that

$$D_\mu^{m_2}X^{i,1}=\sum_{k=0}^{i}a_kX^{k,1}g_k$$

with $\nu_1\left(a_kX^{k,1}\right)\geq\nu_1\left(X^{i,1}\right)$ and $\nu_2(g_k)\geq\nu_2\left(D_\mu^{m_2}\right)$. Thus also for $f\in A$, $X\in\{H_1,F,E,H_2\}$, $i\in\mathbb{N}_0$ we can conclude that there exist $a_k\in K$ and $g_k\in D_q^{m_2}(e,r_2)$ for $0\leq k\leq i$ such that

$$fX^{i,1}=\sum_{k=0}^{i}a_kX^{k,1}g_k \tag{3.3.6}$$

$\nu_1\left(a_kX^{k,1}\right)\geq\nu_1\left(X^{i,1}\right)$ and $\nu_2(g_k)\geq\nu_2\left(f\right)$.
Now let

$$b=\sum_{|\eta|\leq l}a_\eta D_\eta^{m_1}=\sum_{|\eta|\leq l}a_\eta H_1^{\eta_1,1}F^{\eta_2,1}E^{\eta_3,1}H_2^{\eta_4,1}\in B_l.$$

Then equation (3.3.6) implies that for $f\in A$ there exist elements $a_\mu\in K$ and $g_\mu\in D_q^{m_2}(e,r_2)$ for $0\leq|\mu|\leq l$ such that

$$fb=\sum_{|\mu|\leq l}a_\mu D_\mu^{m_1}g_\mu$$

with $\nu_1\left(a_\mu D_\mu\right)\geq\nu_1\left(b\right)\geq 0$ and $\nu_2(g_\mu)\geq\nu_2(f)\geq 0$. Consequently $a_\mu D_\mu\in B_l$ and $g_\mu\in A$ and hence $fb\in B_lA$. This concludes the proof of $AB_l\subseteq B_lA$.

Step 2: We show that $B_lB_k\subseteq B_{l+k}A=AB_{l+k}$.
We will first show the following claim.
Claim 2: For $X,Y\in\{H_1,F,E,H_2\}$ and $i,l\in\mathbb{N}_0$ there exist $a_\mu\in K$ and $g_\mu\in D_q^{m_2}(e,r_2)$ for $0\leq|\mu|<i+l$ such that

$$X^{l,1}Y^{i,1}=Y^{i,1}X^{l,1}+\sum_{|\mu|<i+l}a_\mu D_\mu^{m_1}g_\mu$$

with $\nu_1\left(a_\mu D_\mu^{m_1}\right)\geq\nu_1\left(X^{l,1}Y^{i,1}\right)$ and $\nu_2\left(g_\mu\right)\geq\nu_1\left(X^{l,1}Y^{i,1}\right)$.
Claim 2 is immediate for $X,Y\in\{H_1,H_2\}$ since H_1 and H_2 commute. For

$X \in \{H_1, H_2\}$ and $Y \in \{E, F\}$ or vice versa Claim 2 follows from an equation very similar to (3.3.4) with $g_\mu = 1$ for all μ.
For $X = E, Y = F$. Because we assumed $r_1 - r_2 \geq -\frac{e}{p-1}$ and $r_1 \geq \frac{3e}{p-1} + 2$ in the proposition, we can conclude that $r_1 + r_1 - r_2 - \frac{e}{p-1} \geq 2$. Thus we can use the second part of Lemma 3.2.33 and we can conclude that

$$\begin{aligned} E^{l,1}F^{i,1} = &\sum_{k=1}^{\min\{l,i\}} \sum_{\sigma,\tau} a_{k,\tau,\sigma} F^{i-k,1}E^{l-k,1}H_1^{\tau,2}H_2^{\sigma,2} \\ &+ F^{i,1}E^{l,1} \end{aligned}$$

for some $a_{k,\tau,\sigma}$ with

$$\nu\left(a_{k,\tau,\sigma}\right) - r_1(i-k+l-k) - r_2(\tau+\sigma) \geq -r_1(i+l) + 2.$$

Thus

$$\nu\left(a_{k,\tau,\sigma}\right) + \lfloor r_1 2k \rfloor - \lceil r_2(\tau+\sigma) \rceil \geq 0.$$

Hence $b_{k,\tau,\sigma} := a_{k,\tau,\sigma}\pi^{\lfloor r_1 2k \rfloor}\pi^{-\lceil r_2(\tau+\sigma)\rceil} \in \mathcal{O}$. We then have that

$$a_{k,\tau,\sigma}F^{i-k,1}E^{l-k,1}H_1^{\tau,2}H_2^{\sigma,2} = b_{k,\tau,\sigma}\left[\frac{F^{i-k,1}E^{l-k,1}}{\pi^{\lfloor r_1 2k\rfloor}}\right]\left[\pi^{\lceil r_2(\tau+\sigma)\rceil}H_1^{\tau,2}H_2^{\sigma,2}\right].$$

Since

$$\nu_1\left(\frac{F^{i-k,1}E^{l-k,1}}{\pi^{\lfloor r_1 2k\rfloor}}\right) = -r_1(i+j-2k) - \lfloor r_1 2k \rfloor \geq -r_1(i+j) = \nu_1\left(E^{l,1}F^{i,1}\right)$$

and

$$\nu_2\left(\pi^{\lceil r_2(\tau+\sigma)\rceil}H_1^{\tau,2}H_2^{\sigma,2}\right) = -r_2(\tau+\sigma) + \lceil r_2(\tau+\sigma)\rceil \geq 0 \geq \nu_1\left(E^{l,1}F^{i,1}\right)$$

Claim 2 is proven also in this case. The case where $X = F, Y = E$ is very similar.
Claim 2 together with Step 1 show that for $X_i \in \{H_1, F, E, H_2\}$, $l_i \in \mathbb{N}_0$, $i \in \{1, ..., n\}$, and $\sigma \in S_n$ there exists $a_\mu \in K$ and $g_\mu \in D_q^{m_2}(e,r)$ such that

$$\prod_{i=1}^n X_i^{l_i,1} = \prod_{i=1}^n X_{\sigma(i)}^{l_{\sigma(i)},1} + \sum_{|\mu|<\sum l_i} a_\mu D_\mu^{m_1} g_\mu \tag{3.3.7}$$

with $\nu_1\left(a_\mu D_\mu^{m_1}\right) \geq \sum \nu_1\left(X_i^{l_i,1}\right)$ and $\nu_2(g_\mu) \geq \sum \nu_1\left(X_i^{l_i,1}\right)$. Hence for $b_\mu D_\mu^{m_1} \in B_l$ and $c_\eta D_\eta^{m_1} \in B_k$ there exist $d, e_\alpha \in K$ and $g_\alpha \in D_q^{m_2}(e,r)$ such that

$$b_\mu D_\mu^{m_1} \cdot c_\eta D_\eta^{m_1} = dD_{\mu+\eta}^{m_1} + \sum_{|\alpha|\leq|\mu|+|\eta|} e_\alpha D_\alpha^{m_1} g_\alpha$$

with

$$\begin{aligned}
\nu_1\left(dD^{m_1}_{\mu+\eta}\right) &\geq \nu_1\left(b_\mu D^{m_1}_\mu\right) + \nu_1\left(c_\eta D^{m_1}_\eta\right) \geq 0\\
\nu_1\left(e_\alpha D^{m_1}_\alpha\right) &\geq \nu_1\left(b_\mu D^{m_1}_\mu\right) + \nu_1\left(c_\eta D^{m_1}_\eta\right) \geq 0\\
\nu_2(g_\alpha) &\geq \nu_1\left(b_\mu D^{m_1}_\mu\right) + \nu_1\left(c_\eta D^{m_1}_\eta\right) \geq 0.
\end{aligned}$$

Using also that $|\mu| \leq l$ and $|\eta| \leq k$ we can conclude that $dD^{m_1}_{\mu+\eta} \in B_{l+k}$, $c_\eta D^{m_1}_\eta \in B_{l+k}$ and $g_\alpha \in A$. This implies $b_\mu D^{m_1}_\mu \cdot c_\eta D^{m_1}_\eta \in B_{l+k}A$.
Since

$$B_i = \left\{ \sum_{|\mu|\leq i} a_\mu D^{m_1}_\mu : \nu_1\left(a_\mu D^{m_1}_\mu\right) \geq 0 \right\}$$

Step 2 is proven. Thus $F_lF_k = AB_lB_k \subseteq AB_{l+k}A = AB_{l+k} = F_{l+k}$ and hence property 1 of **3.3.17** is shown.

Now we will prove property 3 of **3.3.17**. Equation (3.3.7) implies that for $\eta, \mu \in \mathbb{N}_0^4$ we have that

$$\left[\pi^{\lceil r_1|\eta|\rceil} D^{m_1}_\eta, \pi^{\lceil r_1|\mu|\rceil} D^{m_1}_\mu\right] \in F_{|\eta|+|\mu|-1}$$

meaning

$$\pi^{\lceil r_1|\eta|\rceil} D^{m_1}_\eta \pi^{\lceil r_1|\mu|\rceil} D^{m_1}_\mu = \pi^{\lceil r_1|\mu|\rceil} D^{m_1}_\mu \pi^{\lceil r_1|\eta|\rceil} D^{m_1}_\eta$$

in $\mathrm{gr}^\bullet B$. Equation (3.3.7) together with Lemma 3.3.15 imply that

$$\left\{\pi^{\lceil r_1|\mu|\rceil} D^{m_1}_\mu : \mu_i \leq p^{m_1}\right\}$$

is a set of generators for $\mathrm{gr}^\bullet B$ as A-algebra. Thus we are left to show that for all $g \in A$ and $\pi^{\lceil r_1|\mu|\rceil} D^{m_1}_\mu$ with $\mu_i \leq p^{m_1}$ we have that

$$\left[g, \pi^{\lceil r_1|\mu|\rceil} D^{m_1}_\mu\right] \in F_{|\mu|-1}$$

i.e. that g and $\pi^{\lceil r_1|\mu|\rceil} D^{m_1}_\mu$ commute in $\mathrm{gr}^\bullet B$. Similar as in the considerations above it suffices to show the following claim.
Claim 3: For $X, Y \in \{H_1, F, E, H_2\}$ there exist $a_k \in K$ and $g_k \in A$ such that

$$X^{l,2}Y^{i,1} = Y^{i,1}X^{l,2} + \sum_{k<i} a_k Y^{k,1} g_k$$

with $\nu_1\left(a_k Y^{k,1}\right) \geq \nu_1\left(Y^{i,1}\right)$ and $\nu_2(g_k) \geq \nu_2\left(X^{l,2}\right)$. For $X, Y \in \{H_1, H_2\}$ this is immediate since H_1 and H_2 commute. For $X = F$ and $Y \in \{H_1, H_2\}$ it follows from equation (3.3.4) and similar for $X = E$, $Y \in \{H_1, H_2\}$. For

$X \in \{H_1, H_2\}$ and $Y = F$ consider

$$\begin{aligned}
H_s^{l,2}F^{i,1} &= F^{i,1}\gamma_l^{m_2}\left(H_s + (-1)^{s+1}i\right)^l \\
&= F^{i,1}H_s^{l,2} + F^{i,1}\sum_{k=0}^{l-1}\frac{l_l^{m_2}!}{l_k^{m_2}!(l-k)!}(-1)^{(s+1)(l-k)}i^{l-k}H_s^{k,2} \\
&= F^{i,1}H_s^{l,2} + F^{i-1,1}\frac{l_i^{m_1}!}{il_{i-1}^{m_1}!}i\sum_{k=0}^{l-1}\frac{l_l^{m_2}!}{l_k^{m_2}!(l-k)!}(-1)^{(s+1)(l-k)}i^{l-k-1}FH_s^{k,2} \\
&= F^{i,1}H_s^{l,2} \\
&+ \pi^{-\lfloor r_1\rfloor}F^{i-1,1}\sum_{k=0}^{l-1}\frac{l_i^{m_1}!}{l_{i-1}^{m_1}!}\frac{l_l^{m_2}!}{l_k^{m_2}!(l-k)!}(-1)^{(s+1)(l-k)}i^{l-k-1}\pi^{\lfloor r_1\rfloor}FH_s^{k,2}
\end{aligned}$$

We can estimate $\nu_1\left(\pi^{-\lfloor r_1\rfloor}F^{i-1,1}\right) = -r_1(i-1) - \lfloor r_1\rfloor \geq -r_1 i = \nu_1\left(F^{i,1}\right)$ and

$$\begin{aligned}
&\nu_2\left(\frac{l_i^{m_1}!}{l_{i-1}^{m_1}!}\frac{l_l^{m_2}!}{l_k^{m_2}!(l-k)!}(-1)^{(s+1)(l-k)}i^{l-k-1}\pi^{\lfloor r_1\rfloor}FH_s^k\right) \\
&\geq \nu_2\left(\frac{\pi^{\lfloor r_1\rfloor}}{l-k}FH_s^k\right) \\
&\geq -\frac{e(l-k)}{p-1} + \lfloor r_1\rfloor - r_2 - r_2k \\
&\geq -\frac{e(l-k)}{p-1} + r_1 - 1 - r_2 - r_2k
\end{aligned}$$

Since we assumed that $r_2 \leq r_1 + \frac{e}{p-1}$ and $r_2 \geq 2 + \frac{3e}{p-1}$ and since $l-k \geq 1$ we can conclude that

$$\begin{aligned}
-\frac{e(l-k)}{p-1} + r_1 - 1 - r_2 - r_2k &\geq -\frac{e(l-k)}{p-1} - \frac{e}{p-1} - r_2k - 1 \\
&\geq -\frac{2e(l-k)}{p-1} - 1 - r_2k \\
&\geq -r_2(l-k) - r_2k = -r_2l = \nu_2\left(H_s^{l,2}\right)
\end{aligned}$$

and thus Claim 3 is true for $X \in \{H_1, H_2\}$ and $Y = F$. Similarly one shows the case $X \in \{H_1, H_2\}$ and $Y = E$. For $X = E$, $Y = F$ Claim 3 follows from equation (3.3.5) and the case $X = F, Y = E$ is very similar and hence we have shown Claim 3.
This shows that the elements $\pi^{\lceil r_1|\mu|\rceil}D_\mu^{m_1}$ are central in $\mathrm{gr}^\bullet B$ and hence $\mathrm{gr}^\bullet B$ is finitely generated by central elements. This shows property 3 of **3.3.17** and hence we proved the Proposition. □

Theorem 3.3.20. *For $4 + \frac{6e}{p-1} < 2r \leq \nu(h)$ the algebra $D_q^\dagger(e,r)$ is a Fréchet-Stein algebra.*

Proof. This is a consequence of **3.3.12**, Proposition 3.3.16 and Proposition

3.3.19. $\square$

3.4 The distribution algebra $D_q(H, K)$

For $4+\frac{6e}{p-1} \leq 2r < \nu(h)$ we denote by $D_q(H,r)$ the K-vector space $C_q^{la}(H,r)'_b$ and by $D_q(H,K)$ the K-vector space $C_q^{la}(H,K)'_b$. Since $C_q^{la}(H,r)'_b$ is a K-Banach Hopf algebra, $D_q(H,r)$ is a K-Banach algebra and since $C_q^{la}(H,K)$ is a complete bornological locally convex K-Hopf algebra, $D_q(H,K)$ is a complete locally convex K-algebra. We will show that there is a finite decomposition

$$D_q(H,r) = \bigoplus_i \delta_{g_i} D_q^{an}(e,r)$$

for certain $\delta_{g_i} \in D_q(H,r)$ that are closely related to the Dirac distributions in the commutative case. We will for $m \in \mathbb{N}_{\geq 3}$ define a subspace

$$D_q^m(H,r) := \oplus_i \delta_{g_i} D_q^m(e,r) \subseteq D_q(H,r)$$

and we will show that it is a K-subalgebra of $D_q(H,r)$. Using the previous section we can conclude that $D_q^m(H,r)$ is Noetherian and that for m_1, m_2, r_1, r_2 as in Proposition 3.3.19 the transition maps

$$D_q^{m_2}(H,r_2) \longrightarrow D_q^{m_1}(H,r_1)$$

are right flat. Since by **3.3.12** there is a sequence (r_n, m_n) such that

$$D_q(H,K) = \varprojlim_{2r<\nu(h)} D_q(H,r) = \varprojlim_n D_q^{m_n}(H,r_n)$$

this implies that $D_q(H,K)$ is a Fréchet Stein algebra.

Remark 3.4.1. Recall that we defined for $g = \begin{pmatrix} g_a & g_b \\ g_c & g_d \end{pmatrix} \in H$ and $\mu \in \mathbb{N}_0^4$ the symbol

$$g^\mu := (a-g_a)^{\mu_1}(b-g_b)^{\mu_2}(c-g_c)^{\mu_3}(d-g_d)^{\mu_4}.$$

Definition 3.4.2. Let $f \in C_q^{la}(H,K)$ and $g \in H$. We then find $r \in \mathbb{Q}$ with $4+\frac{6e}{p-1} < 2r < \nu(h)$ such that $f \in C_q^{la}(H,r)$. Let f_g be the image of f under $C_q^{la}(H,r) \longrightarrow C_q^{an}(g,r)$. It can be uniquely written as

$$f_g = \sum_\mu a_{g,\mu} g^\mu$$

with $a_{g,\mu} \in K$. We then define

$$\delta_g(f) := a_{g,(0,0,0,0)}.$$

This defines a map $\delta_g : C_q^{la}(H,K) \longrightarrow K$. Note that this definition depends on the choice of the ordering of a, b, c, d.

3.4.3. We will denote by ν_r the valuation on $C_q^{la}(H,r)$ as well as the valuation on $C_q^{la}(H,r)'_b$. It will be clear from the context which one is meant. Note that for a covering $H = \coprod_i B(g_i, r)$ we have an equality of vector spaces

$$C_q^{la}(H,r)'_b = \bigoplus_i C_q^{la}(g_i,r)'_b.$$

Then $\nu_r(\lambda) = \min\{\nu_r(\lambda_i)\}$ for $\lambda = \oplus_i \lambda_i \in C_q^{la}(H,r)'_b$.

Lemma 3.4.4. *Let $4 + \frac{6e}{p-1} \leq 2r < \nu(h)$. Then $\delta_g \in C_q^{la}(H,r)'_b$ and $\nu_r(\delta_g) = 0$ for all $g \in H$. Furthermore for $g_1, g_2 \in H$ we have*

$$\nu_r\left(\delta_{g_1}\delta_{g_2} - \delta_{g_1 g_2}\right) \geq \frac{\nu(h)}{2} - r > 0.$$

Proof. The statements $\delta_g \in C_q^{la}(H,r)'_b$ and $\nu_r(\delta_g) = 0$ follow immediately from the definition of the valuation on the dual space. In this proof we will view for $g \in H$ and $\mu \in \mathbb{N}_0^4$ the symbol g^μ as an element in $C_q^{an}(g,r)$. In order to prove that

$$\nu_r\left(\delta_{g_1}\delta_{g_2} - \delta_{g_1 g_2}\right) \geq \frac{\nu(h)}{2} - r$$

we note that by the equation (2.2.1) in the proof of Proposition 2.2.2 we know that the $C_q^{an}(g_1,r)\widehat{\otimes} C_q^{an}(g_2,r)$ part of

$$\Delta\left((g_1 g_2)^\mu\right)$$

can be written as a finite sum with summands of the form

$$\alpha \prod_{i=1}^{k} X_i \otimes \prod_{j=1}^{l} Y_j$$

where $k, l \in \mathbb{N}_0$, $k, l \leq |\mu|$, $k + l \geq |\mu|$, $\alpha \in \mathcal{O}$ and

$$\begin{aligned} X_i &\in \{(a - (g_1)_a), ..., (d - (g_1)_d)\} \subset C_q^{an}(g_1, r); \\ Y_j &\in \{(a - (g_2)_a), ..., (d - (g_2)_d)\} \subset C_q^{an}(g_2, r). \end{aligned}$$

Note that by convention $\prod_\emptyset = 1$. We know that there exist $a_\alpha \in K$ such that $\prod_{i=1}^k X_i = \sum_\alpha a_\alpha g_1^\alpha$. Let $\bar{0} := (0,0,0,0)$. Since $\nu_{r'}$ is multiplicative for all $r' \in \mathbb{Q}$ with $4 + \frac{6e}{p-1} < 2r' < \nu(h)$ we can conclude that

$$\nu\left(a_{\bar{0}}\right) \geq \nu_{r'}\left(\sum_\alpha a_\alpha g_1^\alpha\right) = \nu_{r'}\left(\prod_{i=1}^k X_i\right) = r'k$$

and hence $\nu\left(a_{\bar{0}}\right) \geq \frac{\nu(h)}{2}k$. Analogously we obtain that $\prod_{j=1}^l Y_j = \sum_\beta b_\beta g_2^\beta$ with $\nu\left(b_{\bar{0}}\right) \geq \frac{\nu(h)}{2}l$. Thus we can write $\prod_{i=1}^k X_i \otimes \prod_{j=1}^l Y_j$ as a finite sum of the form

$$\sum_{\alpha,\beta} a_{\alpha,\beta}\left(g_1^{\alpha}\otimes g_2^{\beta}\right)$$

with $a_{\alpha,\beta}=a_\alpha\cdot b_\beta$ and therefore

$$\nu\left(a_{\bar 0,\bar 0}\right)=\nu\left(a_{\bar 0}\cdot b_{\bar 0}\right)\geq\frac{\nu(h)}{2}(k+l)\geq\frac{\nu(h)}{2}|\mu|.$$

Thus

$$\delta_{g_1}\delta_{g_2}\big((g_1,g_2)^{\mu}\big)=\delta_{g_1}\otimes\delta_{g_2}\left(\Delta\big((g_1,g_2)^{\mu}\big)\right)$$

implies

$$\nu\left(\delta_{g_1}\delta_{g_2}\left((g_1g_2)^{\mu}\right)\right)\geq\frac{\nu(h)}{2}|\mu|.$$

Moreover $\delta_{g_1}\delta_{g_2}\left((g_1g_2)^{\mu}\right)=1$ for $\mu=\bar 0$. Because of

$$\delta_{g_1g_2}\big((g_1g_2)^{\mu}\big)=\delta_{0,|\mu|}$$

we obtain

$$\nu\Big([\delta_{g_1}\delta_{g_2}-\delta_{g_1g_2}]\big((g_1g_2)^{\mu}\big)\Big)\geq\frac{\nu(h)}{2}|\mu|.$$

Since

$$\begin{aligned}\nu_r([\delta_{g_1}\delta_{g_2}-\delta_{g_1g_2}])&=\min\left\{\nu\big([\delta_{g_1}\delta_{g_2}-\delta_{g_1g_2}]\left((g_1g_2)^{\mu}\right)\big)-\nu_r\left((g_1g_2)^{\mu}\right):\mu\in\mathbb{N}_0^4\right\}\\&\geq\min\left\{\frac{\nu(h)}{2}|\mu|-r|\mu|:|\mu|\neq 0\right\}\\&=\frac{\nu(h)}{2}-r>0\end{aligned}$$

we are done. □

Lemma 3.4.5. *Let* $e=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. *Then* $\delta_e=1$ *i.e.* $\delta_e\tau=\tau\delta_e=\tau$ *for all* $\tau\in D_q(H,r)$.

Proof. Since $\delta_e=\epsilon$ this follows from the fact that $\left(C_q^{la}(H,r),\Delta,\epsilon\right)$ is a coalgebra. □

Lemma 3.4.6. *The valuation* ν_r *on* $D_q(H,r)=C_q^{la}(H,r)'_b$ *is submultiplicative.*

Proof. Let $\sigma,\tau\in C_q^{la}(H,r)'_b$ and let $m_K:K\widehat{\otimes}K\longrightarrow K$ be the multiplication map. We have that

$$\sigma\tau=m_K\circ(\sigma\otimes\tau)\circ\Delta_r.$$

The map Δ_r is valuation increasing by Proposition 2.2.3. Thus for $f\in C_q^{la}(H,r)$ there exist $f_{1,i},f_{2,i}\in C_q^{la}(H,r)$ such that $\Delta_r(f)=\sum_{i\geq 0}f_{1,i}\otimes f_{2,i}$

and

$$\nu_r\left(f_{1,i}\right) + \nu_r\left(f_{2,i}\right) \geq \nu_r(f).$$

Hence we have that

$$\begin{aligned}
\nu\left(\sigma\tau\left(f\right)\right)) &= \nu\left(\sum_{i\geq 0}\sigma(f_{1,i})\tau(f_{2,i})\right) \\
&\geq \min\left\{\nu\left(\sigma(f_{1,i})\right) + \nu\left(\tau(f_{2,i})\right) : i \geq 0\right\} \\
&\geq \min\left\{\nu_r(\sigma) + \nu_r(f_1, i) + \nu_r(\tau) + \nu_r(f_{2,i}) : i \geq 0\right\} \\
&\geq \min\left\{\nu_r(\sigma) + \nu_r(\tau) + \nu_r(f) : i \geq 0\right\} \\
&= \nu_r(\sigma) + \nu_r(\tau) + \nu_r(f)
\end{aligned}$$

which proves the claim since $f \in C_q^{la}(H, r)$ was arbitrary . $\square$

Lemma 3.4.7. *Let $4 + \frac{6e}{p-1} \leq 2r < \nu(h)$. Then for $g \in H$ we have that*

$$C_q^{an}(g, r)_b' = \delta_g C_q^{an}(e, r)_b' = C_q^{an}(e, r)_b'\delta_g.$$

Moreover for $f \in C_q^{an}(e, r)_b'$ we have that $\nu_r(f) = \nu_r\left(\delta_g f\right) = \nu_r\left(f\delta_g\right)$.

Proof. $\delta_g C_q^{an}(e, r)_b' \subseteq C_q^{an}(g, r)_b'$ follows immediately from the definition of the coproduct on $C_q^{la}(H, r)$. Hence it suffices to show that every $\tau \in C_q^{an}(g, r)_b'$ can be approximated by elements in $\delta_g C_q^{an}(e, r)_b'$. But by Lemma 3.4.5 we know that

$$\tau - \delta_g\delta_{g^{-1}}\tau = (\delta_e - \delta_g\delta_{g^{-1}})\tau.$$

By Lemma 3.4.4

$$\nu_r\left(\delta_e - \delta_g\delta_{g^{-1}}\right) \geq \frac{\nu(h)}{2} - r > 0.$$

Thus the submultiplicativity of the valuation implies

$$\nu_r(\tau - \delta_g\delta_{g^{-1}}\tau) \geq \nu_r(\tau) + \frac{\nu(h)}{2} - r > \nu_r(\tau).$$

Since $\frac{\nu(h)}{2} - r > 0$ is independent of τ the first claim follows.
To show the last claim note that since ν_r is submultiplicative and $\nu_r(\delta_g) = 0$ we have that $\nu_r(\delta_g f) \geq \nu_r(f)$. Assume that $\nu_r(\delta_g f) > \nu_r(f)$. Lemma 3.4.4 and Lemma 3.4.5 imply $\nu_r\left(1 - \delta_{g^{-1}}\delta_g\right) > 0$. Thus

$$\begin{aligned}
\nu_r(f) &= \nu_r(f - \delta_{g^{-1}}\delta_g f + \delta_{g^{-1}}\delta_g f) \\
&= \nu_r((1 - \delta_{g^{-1}}\delta_g)f + \delta_{g^{-1}}\delta_g f) \\
&\geq \min\{\nu_r(1 - \delta_{g^{-1}}\delta_g) + \nu_r(f), \nu_r(\delta_{g^{-1}}) + \nu_r(\delta_g f)\} \\
&> \nu_r(f)
\end{aligned}$$

which cannot be true. $\square$

Corollary 3.4.8. *Let* $4 + \frac{6e}{p-1} \leq 2r < \nu(h)$ *and* $H = \coprod_{i \in I} B(g_i, r)$. *Then*

$$D_q(H,r) = \bigoplus_i \delta_{g_i} D_q^{an}(e,r)$$

and $\delta_{g_i} D_q^{an}(e,r) \cdot \delta_{g_j} D_q^{an}(e,r) = \delta_{g_i g_j} D_q^{an}(e,r)$.

Proof. We have that

$$D_q(H,r) = C_q^{la}(H,r)'_b = \left(\bigoplus_i C_q^{an}(g_i,r) \right)'_b = \bigoplus_i C_q^{an}(g_i,r)'_b = \bigoplus_i \delta_{g_i} C_q^{an}(e,r)'_b.$$

Since by Lemma 3.3.10 we know that $C_q^{an}(e,r)'_b = D_q^{an}(e,r)$ we have proven the Corollary. $\square$

Definition 3.4.9. We define for $4 + \frac{6e}{p-1} \leq 2r < \nu(h)$ the K-vector space

$$D_q^m(H,r) := \bigoplus_i \delta_{g_i} D_q^m(H,r)$$

We will show that $D_q^m(H,r)$ is a subalgebra of $D_q(H,r)$. In order to reach this goal, we need some auxiliary Lemmas.

Lemma 3.4.10. *Let* $4 + \frac{6e}{p-1} \leq 2r < \nu(h)$. *Let for* $\mu \in \mathbb{N}_0^4$ *the element* $b_\mu \in D_q^{an}(e,r)$ *be defined by* $b_\mu(e^\eta) = \delta_{\mu,\eta}$. *Let* $g \in H$. *By Lemma 3.4.7 there exists a sequence* $(a_\eta)_{\eta \in \mathbb{N}^4}$ *with* $a_\eta \in K$ *such that*

$$b_\mu \delta_g = \delta_g \sum_\eta a_\eta b_\eta.$$

Then $a_\eta = 0$ *for all* η *with* $|\eta| < |\mu|$ *and*

$$\nu_r(a_\eta b_\eta) \geq \left(\frac{\nu(h)}{2} - r \right) (|\eta| - |\mu|) + \nu_r(b_\mu)$$

for all $\eta \in \mathbb{N}^4$. *Moreover* $\nu_r(b_\mu) = \nu_r\left(\sum_\eta a_\eta b_\eta\right)$.

Proof. **Step 1:** We have that

$$\nu(\delta_g b_\mu(g^\eta)) \geq (|\eta| - |\mu|) \frac{\nu(h)}{2}$$

for all $\eta \in \mathbb{N}^4$ and $\delta_g b_\mu(g^\eta) = 0$ if $|\eta| < |\mu|$. Analogously

$$\nu(b_\mu \delta_g(g^\eta)) \geq (|\eta| - |\mu|) \frac{\nu(h)}{2}$$

and $\delta_g b_\mu(g^\eta) = 0$ if $|\eta| < |\mu|$.

We will only show the statement for $\delta_g b_\mu$ since the other one is obtained in the same way. Note that

$$\delta_g b_\mu(g^\eta) = \delta_g \otimes b_\mu(\Delta(g^\eta)).$$

As in the proof of Lemma 3.4.4 we can write the $C_q^{an}(g,r)\widehat{\otimes}C_q^{an}(e,r)$ part of $\Delta\left(g^\eta\right)$ as a sum with summands of the form

$$\alpha\prod_{i=1}^{k}X_i\otimes\prod_{j=1}^{l}Y_j$$

where $k,l\in\mathbb{N}_0$, $k,l\leq|\eta|$, $k+l\geq|\eta|$, $\alpha\in\mathcal{O}$ and

$$\begin{aligned}X_i&\in\left\{(a-g_a),...,(d-g_d)\right\}\subset C_q^{an}(g,r);\\ Y_j&\in\left\{(a-1),b,c,(d-1)\right\}\subset C_q^{an}(e,r).\end{aligned}$$

We can write

$$\alpha\prod_{i=1}^{k}X_i\otimes\prod_{j=1}^{l}Y_j=\sum_{|\beta|\leq k,|\gamma|\leq l}\alpha_{\beta,\gamma}g^\beta\otimes e^\gamma.$$

Because of $|\gamma|\leq l\leq|\eta|$ in the sum we see that $\delta_g b_\mu\left(g^\eta\right)=0$ if $|\eta|<|\mu|$. In the proof of Lemma 3.4.4 we showed that $\nu(\alpha_{\bar{0},\bar{0}})\geq\frac{\nu(h)}{2}|\eta|$ for $\bar{0}=(0,0,0,0)$. With the same strategy one can show that

$$\nu\left(\alpha_{\beta,\gamma}\right)\geq\frac{1}{2}\nu(h)(k+l-|\beta|-|\gamma|).$$

Since we are interested in $\left(\delta_g\otimes b_\mu\right)\left(\Delta\left(g^\mu\right)\right)$ only the term $\alpha_{\bar{0},\mu}$ contributes. Because of

$$\nu\left(\alpha_{\bar{0},\mu}\right)\geq\frac{1}{2}\nu(h)(k+l-|\mu|)\geq\frac{1}{2}\nu(h)(|\eta|-|\mu|)$$

Step 1 is proved.

Step 2: Let $c_\mu\in C_q^{an}(g,r)_b'$ for $\mu\in\mathbb{N}_0^4$ be defined by $c_\mu\left(g^\eta\right):=\delta_{\mu,\eta}$. Then

$$\begin{aligned}\delta_g b_\mu&=\sum_\eta\left(\delta_g b_\mu\right)\left(g^\eta\right)c_\eta;\\ g_\mu\delta_g&=\sum_\eta\left(b_\mu\delta_g\right)\left(g^\eta\right)c_\eta.\end{aligned}$$

By Step 1 the sum $\sum_\eta\left(\delta_g b_\mu\right)\left(g^\eta\right)c_\eta$ converges in $C_q^{an}(e,r)_b'$. Since the K-vector space $\left\{\sum_{|\alpha|\leq k}a_\alpha g^\alpha:k\in\mathbb{N}_0\right\}$ is dense in $C_q^{an}(g,r)$, Step 2 follows from $\delta_g b_\mu\left(g^\alpha\right)=\left(\sum_\eta\left(\delta_g b_\mu\right)\left(g^\eta\right)c_\eta\right)\left(g^\alpha\right)$.

Step 3: Let $b=\sum_{|\eta|=|\mu|}a_\eta b_\eta$ with $\min\{\nu(a_\eta)\}=0$. Then

$$b\delta_g=\sum_\eta f_\eta c_\eta$$

with

$$\min\{\nu\left(f_\eta\right) : |\eta| = |\mu|\} = 0$$

and $f_\eta = 0$ if $|\eta| < |\mu|$. The analogous statement is true for $\delta_g b$.
That $f_\eta = 0$ if $|\eta| < |\mu|$ follows directly from Step 1 and Step 2. We will now focus on the first statement of Step 3.
Note that $\nu_r\left(\sum_\eta f_\eta c_\eta\right) = \min\{\nu(f_\eta) - r|\eta|\}$. Since $\nu_r\left(b\delta_g\right) \geq -r|\mu| + \nu_r(\delta_g)$ and $\nu_r(\delta_g) = 0$ this implies

$$\min\{\nu\left(f_\eta\right) : |\eta| = |\mu|\} \geq 0.$$

Assume

$$\min\{\nu\left(f_\eta\right) : |\eta| = |\mu|\} = c > 0.$$

Step 1 and Step 2 then imply that

$$\nu_r\left(b\delta_g\right) \geq \min\left\{c - r|\mu|, (|\eta| - |\mu|)\frac{\nu(h)}{2} - r|\eta| : |\eta| > |\mu|\right\} > -r|\mu|.$$

Because of the submultiplicativity of the valuation it follows then that

$$\nu_r\left(b\delta_g\delta_{g^{-1}}\right) > -|\mu|r = \nu_r(b).$$

But also $\nu_r\left(b\left(1 - \delta_g\delta_{g^{-1}}\right)\right) > -|\mu|r$ by Lemma 3.4.4. Because of

$$b = b\delta_g\delta_{g^{-1}} + b\left(1 - \delta_g\delta_{g^{-1}}\right)$$

we thus obtain that $\nu_r(b) > -|\mu|r = \nu_r(b)$ which is not true.

Step 4: We are now ready to prove the Lemma. Consider

$$b_\mu\delta_g = \sum_{|\eta|=|\mu|} \delta_g a_\eta b_\eta + \sum_{|\eta|>|\mu|} \delta_g a_\eta b_\eta. \tag{3.4.1}$$

By Step 3 we know that $b_\mu\delta_g = \sum_{|\eta|\geq|\mu|} f_\eta c_\eta$ with $\min\{\nu(f_\eta) : |\eta| = |\mu|\} = 0$. Using Step 3 for the right hand side of equation (3.4.1) we can conclude that

$$\sum_{|\eta|=|\mu|} \delta_g a_\eta b_\eta + \sum_{|\eta|>|\mu|} \delta_g a_\eta b_\eta = \sum_{|\eta|\geq|\mu|} g_\eta c_\eta$$

with $\min\{\nu(g_\eta) : |\eta| = |\mu|\} = \min\{\nu(a_\eta) : |\eta| = |\mu|\}$. Comparing both sides we can conclude that

$$\min\{\nu(a_\eta) : |\eta| = |\mu|\} = 0.$$

We show by induction that

$$\min\{a_\eta : |\eta| = |\mu| + n\} \geq n\frac{\nu(h)}{2} \tag{3.4.2}$$

The case $n = 0$ is already done. Assume that the statement is true for all $k \leq n$. From Step 1-3 we can extract the following. We have

$$b_\mu \delta_g = \sum_{|\eta| \geq |\mu|} f_\eta c_\eta$$

with $\nu(f_\eta) \geq (|\eta| - |\mu|)\frac{\nu(h)}{2}$. By Step 1 and 2 and by the induction hypothesis we have for $|\mu| \leq |\eta| \leq |\mu| + n$ that

$$\delta_g a_\eta b_\eta = \sum_{|\alpha| \geq |\eta|} u_{\alpha,\eta} c_\alpha$$

with

$$\begin{aligned}
\nu\left(u_{\alpha,\eta}\right) &\geq \nu(a_\eta) + (|\alpha| - |\eta|)\frac{\nu(h)}{2} \\
&\geq (|\eta| - |\mu|)\frac{\nu(h)}{2} + (|\alpha| - |\eta|)\frac{\nu(h)}{2} \\
&= (|\alpha| - |\mu|)\frac{\nu(h)}{2}
\end{aligned}$$

Thus

$$\begin{aligned}
\delta_g \sum_{|\eta| \geq |\mu|+n+1} a_\eta b_\eta &= b_\mu \delta_g - \delta_g \sum_{|\mu|+n \geq |\eta| \geq |\mu|} a_\eta b_\eta \\
&= \sum_{|\eta| \geq |\mu|} \sum_{|\alpha| \geq |\eta|} (f_\eta - u_{\alpha,\eta}) c_\eta
\end{aligned} \tag{3.4.3}$$

Where $\sum_{|\alpha| \geq |\eta|}(f_\eta - u_{\alpha,\eta}) = 0$ for $|\eta| \leq |\mu| + n$ by Step 1 and 2 and hence the right hand side starts at $|\eta| = |\mu| + n + 1$. Moreover for $|\eta| \geq |\mu| + n + 1$ we can conclude that

$$\nu\left(\sum_{|\alpha| \geq |\eta|} (f_\eta - u_{\alpha,\eta})\right) \geq \min\{\nu(f_\eta), \nu(u_{\alpha,\eta})\} \geq (|\eta| - |\mu|)\frac{\nu(h)}{2}.$$

Step 3 and the estimates for the terms in equation (3.4.3) imply now that

$$\begin{aligned}
\min\{a_\eta : |\eta| = |\mu| + n + 1\} &= \min\left\{\nu\left(\sum_{|\alpha| \geq |\eta|} (f_\eta - u_{\alpha,\eta})\right) : |\eta| = |\mu| + n + 1\right\} \\
&\geq (n+1)\frac{\nu(h)}{2}
\end{aligned}$$

and we have shown (3.4.2). Thus

$$\begin{aligned}
\nu_r\left(a_\eta b_\eta\right) &= \nu(a_\eta) + \nu_r(b_\eta) \\
&\geq \frac{\nu(h)}{2}(|\eta| - |\mu|) - r|\eta| \\
&= \left(\frac{\nu(h)}{2} - r\right)(|\eta| - |\mu|) - r|\mu| \\
&= \left(\frac{\nu(h)}{2} - r\right)(|\eta| - |\mu|) + \nu_r(b_\mu) > \nu_r(b_\mu).
\end{aligned}$$

Since $\min\{a_\eta : |\eta| = |\mu|\} = 0$ we also obtain $\nu_r(b_\mu) = \nu_r\left(\sum_\eta a_\eta b_\eta\right)$. $\square$

Lemma 3.4.11. *Let $4 + \frac{6e}{p-1} \leq 2r < \nu(h)$. There exists $\epsilon_1 > 0$ such that for $\alpha \in \mathbb{N}_0^4$ there exist $a_\beta \in K$ with*

$$b_\alpha = \sum_{\beta \in \mathbb{N}_0^4} a_\beta \bar{D}_\beta$$

and

$$\nu_r\left(a_\beta \bar{D}_\beta\right) \geq \nu_r(b_\alpha) + \epsilon_1|\beta - \alpha|$$

for all $\beta \in \mathbb{N}_0^4$.

Proof. **Sublemma 1:** There exists $\epsilon_1 > 0$ such that for $\gamma \in \mathbb{N}_0^4$ there exist a $\lambda \in \mathcal{O}^\times$ and $c_\delta \in K$ such that

$$b_\gamma - \lambda\bar{D}_\gamma = \sum_\delta c_\delta b_\delta,$$

$c_\gamma = 0$ and

$$\nu_r(c_\delta b_\delta) \geq \epsilon_1(|\delta - \gamma|) + \nu_r(b_\gamma)$$

In particular we have that $\nu_r\left(b_\gamma - \lambda\bar{D}_\gamma\right) \geq \epsilon_1 + \nu_r(b_\gamma)$ and hence we know $\nu_r\left(b_\gamma\right) = \nu_r\left(\lambda\bar{D}_\gamma\right)$.

Proof of Sublemma 1. Let $\lambda = \left(\left\langle \bar{D}_\gamma, e^\gamma \right\rangle\right)^{-1}$. By Lemma 3.2.24/4 we know that $\nu(\lambda) = 0$. Hence Theorem 3.2.25 implies that there exists $\epsilon > 0$ such that

$$\nu\left((b_\gamma - \lambda\bar{D}_\gamma)\left(e^\delta\right)\right) \geq -\lfloor r|\gamma|\rfloor + \lfloor r|\delta|\rfloor + 1 + \epsilon$$

for all $\gamma, \delta \in \mathbb{N}_0^4$. Thus we can conclude

$$\begin{aligned}
\nu\left((b_\gamma - \lambda\bar{D}_\gamma)\left(e^\delta\right)\right) - \nu_r\left(e^\delta\right) &\geq -\lfloor r|\gamma|\rfloor + \lfloor r|\delta|\rfloor + 1 + \epsilon - r|\delta| \\
&\geq \nu_r(b_\gamma) + 1 + \epsilon - 1 \\
&\geq \epsilon + \nu_r\left(b_\gamma\right)
\end{aligned} \tag{3.4.4}$$

By Theorem 3.2.25 we also know that

$$\nu\left(\left(b_\gamma-\lambda\bar{D}_\gamma\right)\left(e^\delta\right)\right)\geq\frac{1}{6}|\gamma-\delta|-\lfloor r|\gamma|\rfloor+\lfloor r|\delta|\rfloor.$$

Thus we can conclude that

$$\begin{aligned}\nu\left(\left(b_\gamma-\lambda\bar{D}_\gamma\right)\left(e^\delta\right)\right)-\nu_r\left(e^\delta\right)&\geq\frac{1}{6}|\gamma-\delta|-\lfloor r|\gamma|\rfloor+\lfloor r|\delta|\rfloor-r|\delta|\\&\geq\frac{1}{6}|\gamma-\delta|-1+\nu_r(b_\gamma).\end{aligned}\tag{3.4.5}$$

Combining (3.4.4) and (3.4.5) we can find ϵ_1 independent of δ,γ such that the inequality in Sublemma 1 is fulfilled for $c_\delta:=\left(b_\gamma-\lambda\bar{D}_\gamma\right)\left(e^\delta\right)$.

Sublemma 2: There exists a set $\{a_{\beta,n}:a_{\beta,n}\in K,\ n\in\mathbb{N},\ \beta\in\mathbb{N}_0^4\}$ such that

1. $\nu_r\left(a_{\beta,n}\bar{D}_\beta\right)\geq\epsilon_1|\beta-\mu|+\nu_r(b_\mu)$.
2. $b_\mu-\sum_\beta a_{\beta,n}\bar{D}_\beta=\sum_\gamma d_{\gamma,n}b_\gamma$ for some $d_{\gamma,n}\in K$ with

$$\nu_r\left(d_{\gamma,n}b_\gamma\right)\geq\epsilon_1|\beta-\mu|+(n-1)\epsilon_1+\nu_r(b_\mu).$$

3. $\nu_r\left(b_\mu-\sum_\beta a_{\beta,n}\bar{D}_\beta\right)\geq n\epsilon_1$.
4. $\nu_r\left(a_{\beta,n}\bar{D}_\beta-a_{\beta,n-1}\bar{D}_\beta\right)\geq\epsilon_1$.

Proof of Sublemma 2. We can construct such a set inductively. For $n=1$ we can choose $a_{\beta,1}=0$ if $\beta\neq\mu$ and $a_{\mu,1}=\left(\left\langle\bar{D}_\mu,e^\mu\right\rangle\right)^{-1}$ by Sublemma 1. If we have a for $N\in\mathbb{N}$ a set $\{a_{\beta,n}:a_{\beta,n}\in K,\ n\leq N,\ \beta\in\mathbb{N}_0^4\}$ which fulfills 1.-4. then we choose $a_{\beta,N+1}=a_{\beta,N}+d_{\beta,N}\left(\left\langle\bar{D}_\beta,e^\beta\right\rangle\right)^{-1}$.
By Sublemma 1 the set $\{a_{\beta,n}:a_{\beta,n}\in K,\ n\leq N+1,\ \beta\in\mathbb{N}_0^4\}$ fulfills 1.-4. since $\{a_{\beta,n}:a_{\beta,n}\in K,\ n\leq N,\ \beta\in\mathbb{N}_0^4\}$ fulfills 1.-4. and thus Sublemma 2 is proven.

Sublemma 2/4 implies that $\lim\limits_{n\to\infty}a_{\beta,n}$ exists and Sublemma 2/3 implies that for $a_\beta:=\lim\limits_{n\to\infty}a_{\beta,n}$ we have $\sum_\beta a_\beta\bar{D}_\beta=b_\mu$. The estimates in the Lemma follow from the estimates in Sublemma 2. □

Corollary 3.4.12. *Let $4+\frac{6e}{p-1}\leq 2r<\nu(h)$. For $\mu\in\mathbb{N}_0^4$ there exist $a_\gamma\in K$ for $\gamma\in\mathbb{N}_0^4$ such that*

$$D_\mu\delta_g=\delta_g\sum_\gamma a_\gamma D_\gamma$$

and $\nu_r\left(a_\gamma D_\gamma\right)\geq\nu_r\left(D_\mu\right)$ for all $\gamma\in\mathbb{N}_0$. Moreover there exists a $c>0$ independent of μ such that for $|\gamma|>|\mu|$ we have that

$$\nu_r\left(a_\gamma D_\gamma\right)\geq c(|\gamma|-|\mu|)+\nu_r\left(D_\mu\right).$$

Proof. We will show the Lemma first for $\bar{D}_\mu := \binom{H_1}{\mu_1}\frac{E^{\mu_2}F^{\mu_3}}{\mu_2!\mu_3!}\binom{H_2}{\mu_4}$ instead of D_μ.

We can write $\bar{D}_\mu = \sum_\eta d_\eta b_\eta$ where $d_\eta = \bar{D}_\mu(e^\eta)$. Using the same argument as in Sublemma 1 of Lemma 3.4.11 we can conclude that there exists $\epsilon_1 > 0$ independent of μ such that for any $\eta \in \mathbb{N}_0^4$ we have

$$\nu_r(d_\eta b_\eta) \geq \nu_r(\bar{D}_\mu) + \epsilon_1|\eta - \mu|.$$

Thus using $|\eta - \mu| \geq \big||\eta| - |\mu|\big|$ and Lemma 3.4.10 we can conclude that there exist $f_\alpha^\eta \in K$ such that

$$\bar{D}_\mu\delta_g = \sum_\eta d_\eta b_\eta \delta_g = \delta_g \sum_\eta \sum_\alpha d_\eta f_\alpha^\eta b_\alpha$$

and

$$\nu_r\left(d_\eta f_\alpha^\eta b_\alpha\right) \geq \nu_r\left(\bar{D}_\mu\right) + \min\left\{\epsilon_1, \frac{\nu(h)}{2} - r\right\}\big||\alpha| - |\mu|\big|.$$

Let $b_\alpha = \sum_\beta e_\beta \bar{D}_\beta$. Because of Lemma 3.4.11 we obtain

$$\nu_r\left(e_\beta \bar{D}_\beta\right) \geq \nu_r(b_\alpha) + \epsilon_1|\beta - \alpha|$$

for all $\beta \in \mathbb{N}_0^4$. Combing with the previous considerations we can find $f_\gamma \in K$ such that

$$\bar{D}_\mu\delta_g = \delta_g \sum_\gamma f_\gamma \bar{D}_\gamma$$

and

$$\nu_r\left(f_\gamma \bar{D}_\gamma\right) \geq \nu_r(\bar{D}_\mu) + \min\left\{\epsilon_1, \frac{\nu(h)}{2} - r\right\}||\gamma| - |\mu||$$

for $\gamma \in \mathbb{N}_0^4$. Thus we showed the Lemma for $\bar{D}$.

By Lemma 3.3.5 there exist $u_\delta \in K$ such that

$$D_\mu = \sum_{|\delta| \leq |\mu|} u_\delta \bar{D}_\delta$$

with $\nu_r\left(u_\delta \bar{D}_\delta\right) \geq \nu_r\left(D_\mu\right)$. Lemma 3.3.5 also implies that for $\gamma \in \mathbb{N}_0^4$ there exist $v_{\gamma,\iota} \in K$ such that

$$\bar{D}_\gamma = \sum_{|\iota| \leq |\gamma|} v_{\gamma,\iota} D_\iota$$

with $\nu_r\left(v_{\gamma,\iota} D_\delta\right) \geq \nu_r\left(\bar{D}_\gamma\right)$. Hence we can conclude that

$$\begin{aligned}D_\mu\delta_g &= \sum_{|\delta| \leq |\mu|} u_\delta \bar{D}_\delta \delta_g = \delta_g \sum_{|\delta| \leq |\mu|} \sum_\gamma u_\delta f_{\delta,\gamma} \bar{D}_\gamma \\ &= \delta_g \sum_{|\delta| \leq |\mu|} \sum_\gamma \sum_{|\iota| \leq |\gamma|} u_\delta f_{\delta,\gamma} v_{\delta,\gamma,\iota} D_\iota.\end{aligned}$$

By the estimates above we have that

$$\nu_r\left(u_\delta f_{\delta,\gamma} v_{\delta,\gamma,\iota} D_\iota\right) \geq \nu_r\left(D_\mu\right)$$

for all $\iota \in \mathbb{N}_0^4$ and

$$\nu_r\left(u_\delta f_{\delta,\gamma} v_{\delta,\gamma,\iota} D_\iota\right) \geq \nu_r\left(D_\mu\right) + c(|\iota| - |\mu|)$$

for all $|\iota| > |\mu|$ and $c := \min\left\{\epsilon_1, \frac{\nu(h)}{2} - r\right\}$. ☐

3.4.13. For the rest of the section we will abbreviate $\nu_{D_q^m(e,r)}$ by $\nu_{m,r}$.

Corollary 3.4.14. *Let $4 + \frac{6e}{p-1} \leq 2r < \nu(h)$. Let $c > 0$ be as in Corollary 3.4.12 and let $m \in \mathbb{N}$ be such that $cp^m \geq 2$.*
For $i \leq p^m$ and $X \in \{H_1, H_2, F, E\}$ we have

$$\frac{X^i}{i!}\delta_g = \delta_g \sum_\eta a_\eta D_\eta$$

with $\sum_\eta a_\eta D_\eta \in D_q^m(e,r)$ and $\nu_{m,r}\left(\sum_\eta a_\eta D_\eta\right) \geq \nu_{m,r}\left(\frac{X^i}{i!}\right)$.

Proof. For $|\eta| < pp^m$ we have that $\nu_r(D_\mu) = \nu_{m,r}(D_\mu)$. Thus Lemma 3.4.12 implies that $\nu_{m,r}(a_\eta D_\eta) \geq \nu_{m,r}(\frac{X^i}{i!})$ for $|\eta| < pp^m$. In general we know that

$$\nu_{m,r}(D_\eta) = \nu_r(D_\eta) - \nu\left(l_{\eta_1}^m! l_{\eta_2}^m! l_{\eta_3}^m! l_{\eta_4}^m!\right) \geq \nu_r(D_\eta) - \nu\left(l_{|\eta|}^m!\right).$$

For $k \in \mathbb{N}_0$ we have $\nu(l_k^m!) \leq \frac{l_k^m}{p-1} \leq l_k^m$ and thus Lemma 3.4.12 implies

$$\begin{aligned}\nu_{m,r}\left(a_\eta D_\eta\right) &\geq \nu_r(a_\eta D_\eta) - \nu\left(l_{|\eta|}^m!\right) \\ &\geq \nu_r\left(\frac{X^i}{i!}\right) + c(|\eta| - i) - l_{|\eta|}^m.\end{aligned}$$

Since $|\eta| \geq p^m l_{|\eta|}^m$, $i \leq p^m$ and $cp^m \geq 2$ we can conclude for $|\eta| \geq pp^m$ that

$$\begin{aligned}\nu_{m,r}\left(a_\eta D_\eta\right) &\geq \nu_r\left(\frac{X^i}{i!}\right) + c(|\eta| - i) - l_{|\eta|}^m \\ &\geq \nu_{m,r}\left(\frac{X^i}{i!}\right) + cp^m(l_{|\eta|}^m - 1) - l_{|\eta|}^m \\ &\geq \nu_{m,r}\left(\frac{X^i}{i!}\right) + 2(l_{|\eta|}^m - 1) - l_{|\eta|}^m \\ &\geq \nu_{m,r}\left(\frac{X^i}{i!}\right) + l_{|\eta|}^m - 2. \end{aligned} \tag{3.4.6}$$

Because $p \geq 2$ and $|\eta| \geq pp^m$ we obtain $l_{|\eta|}^m \geq 2$ and thus

$$\nu_{m,r}\left(a_\eta D_\eta\right) \geq \nu_{m,r}\left(\frac{X^i}{i!}\right).$$

Since $l_{|\eta|}^m \to \infty$ as $|\eta| \to \infty$ the estimate (3.4.6) implies $\sum_\eta a_\eta D_\eta \in D_q^m(e,r)$. Furthermore (3.4.6) shows that

$$\nu_{m,r}\left(\sum_\eta a_\eta D_\eta\right) \geq \nu_{m,r}\left(\frac{X^i}{i!}\right).$$

□

Corollary 3.4.15. *Let $4 + \frac{6e}{p-1} \leq 2r < \nu(h)$. Let $c > 0$ be as in Corollary 3.4.12 and let $m \in \mathbb{N}$ be such that $cp^m \geq 2$. Then*

$$\delta_g D_q^m(e,r) = D_q^m(e,r)\delta_g.$$

Moreover for $A, B \in D_q^m(e,r)$ with $A\delta_g = \delta_g B$ we have $\nu_{m,r}(A) = \nu_{m,r}(B)$.

Proof. By Lemma 3.3.15 D_μ^m can be written up to an element in $\mathcal{O}^\times$ as a product of elements $\frac{X^i}{i!}$ with $X \in \{H_1, H_2, F, E\}$ and $i \leq p^m$ in such a way that the valuation of D_μ^m is the sum of the valuations of these factors.
Using Corollary 3.4.14 and the submultiplicativity of $\nu_{m,r}$ we can conclude that there exist $a_\eta \in K$ with $D_\mu^m \delta_g = \delta_g \sum_\eta a_\eta D_\eta^m$ and

$$\sum_\eta a_\eta D_\eta^m \in D^m(e,r)$$

and $\nu_{m,r}\left(\sum_\eta a_\eta D_\eta\right) \geq \nu_{m,r}(D_\mu)$. Hence for $D_q^m(e,r)\delta_g \subseteq \delta_g D_q^m(e,r)$ and for $A\delta_g = \delta_g B$ we obtain $\nu_{m,r}(B) \geq \nu_{m,r}(A)$.
The other inclusion resp. inequality is obtained analogously by using statements analogous to Corollary 3.4.12 and Corollary 3.4.14. □

Proposition 3.4.16. *Let $4 + \frac{6e}{p-1} \leq 2r < \nu(h)$ and let $H = \coprod_i B(g_i, r)$. Let $c > 0$ be as in Corollary 3.4.12 and let $m \in \mathbb{N}$ be such that $cp^m \geq 2$. The K-vector space*

$$D_q^m(H,r) = \oplus_i \delta_{g_i} D^m(e,r)$$

is a subalgebra of $D_q(H,r)$. It is a K-Banach algebra with respect to the valuation ν_r^m given by $\nu_r^m(\oplus \delta_{g_i} f_i) := \min\{\nu_{m,r}(f_i)\}$. Moreover $D_q^m(H,r)$ is Noetherian.

Proof. The first statement follows from Corollary 3.4.15. By Proposition 3.3.16 $D_q^m(e,r)$ is Noetherian. Since $D_q^m(H,r)$ is a finite module over $D_q^m(e,r)$ it is Noetherian as well. □

Proposition 3.4.17. *Let $c > 0$ be as in Corollary 3.4.12 and let $m \in \mathbb{N}$ be such that $cp^m \geq 2$. Let $r_1, r_2 \in \frac{\mathbb{Z}}{p^m}$ be such that $4 + \frac{6e}{p-1} \leq 2r_1 < 2r_2 < \nu(h)$ and $r_2 \leq r_1 + \frac{e}{p-1}$ and let $m \leq m_1 < m_2$ be such that $D_q^{m_2}(e, r_2) \subseteq D_q^{m_1}(e, r_1)$.*

Then the K-Banach algebra morphism

$$D_q^{m_2}(H, r_2) \longrightarrow D_q^{m_1}(H, r_1)$$

is right flat with dense image.

Proof. By our assumption on m the spaces $D_q^{m_1}(H, r_1)$ and $D_q^{m_2}(H, r_2)$ are K-algebras and the inclusion is an inclusion of K-algebras. We know by Proposition 3.3.19 that the maps

$$D_q^{m_2}(e, r_2) \to D_q^{m_1}(e, r_1)$$

are right flat with dense image. Tensoring over $D_q^{m_2}(e, r_2)$ with $D_q^{m_2}(H, r_2)$ we obtain that the map

$$D_q^{m_2}(H, r_2) \otimes_{D_q^{m_2}(e,r_2)} D_q^{m_2}(e, r_2) \longrightarrow D_q^{m_2}(H, r_2) \otimes_{D_q^{m_2}(e,r_2)} D_q^{m_1}(e, r_1)$$

is right flat with dense image. Since

$$\begin{aligned} D_q^{m_2}(H, r_2) \otimes_{D_q^{m_2}(e,r_2)} D_q^{m_1}(H, r_1) &\cong \left(\bigoplus_i \delta_{g_i} D_q^{m_2}(e, r_2) \right) \otimes_{D_q^{m_2}(e,r_2)} D_q^{m_1}(e, r_1) \\ &\cong \bigoplus_i \left(\delta_{g_i} D_q^{m_2}(e, r_2) \otimes_{D_q^{m_2}(e,r_2)} D_q^{m_1}(e, r_1) \right) \\ &\cong \bigoplus_i \delta_{g_i} D_q^{m_1}(e, r_1) \\ &= D_q^{m_1}(H, r_1) \end{aligned}$$

the Lemma is proven. □

Theorem 3.4.18. *Let $\nu(h) > 4 + \frac{6e}{p-1}$ and let $q = e^h$. Then $D_q(H, K)$ is a Fréchet-Stein algebra.*

Proof. Note that by the last statement of Lemma 3.4.7 and by **3.4.3** we have for $f = \oplus_i \delta_{g_i} f_i \in D_q(H, r)$ that $\nu_r(f) = \min\{\nu_r(f_i)\}$. Thus by Proposition 3.4.16 and **3.3.12** there is a sequence $(r_n, m_n)_{n\in\mathbb{N}}$ such that the systems of K-Banach spaces $\left\{D_q^{m_n}(H, r_n) : n \in \mathbb{N}\right\}$ and $\{D_q(H, r) : 2r < \nu(h)\}$ are cofinal. We can assume that (r_n, m_n) and (r_{n+1}, m_{n+1}) fulfill the requirements of Proposition 3.4.17 by enlarging m_n, m_{n+1} if necessary. Hence we can assume that the systems are cofinal systems of K-Banach algebras and thus

$$D_q(H, K) = \varprojlim_{2r<\nu(h)} D_q(H, r) = \varprojlim_n D_q^{m_n}(H, r_n).$$

Then Proposition 3.4.17 and Proposition 3.4.16 imply that $D_q(H, K)$ is a locally convex projective limit of Noetherian Banach algebras with right flat transition maps i.e a Frechet-Stein algebra. □

Chapter 4

The quantum p-adic upper half plane

In this chapter we fix a $q \in \mathcal{O}^\times - \{1, -1\}$ which fulfills $\nu(1-q) \geq 2$. All radii r in this chapter are elements of $\mathbb{R}$ and in contrast to the previous chapters we do not assume $r \in \mathbb{Q}$.

4.1 Overview of the construction

We will shortly recall some features of the classical p-adic upper half plane and then explain our strategy for the construction of a quantized analogue. For a more detailed discussion of the p-adic upper half plane see e.g. [DT08] or [ST97].

The p-adic upper half plane $\mathcal{H}$ is a rigid analytic space with $\mathbb{C}_p$-valued points $\mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(K)$. Let $\mathcal{T}$ be the Bruhat-Tits tree associated to $\mathrm{PGL}(2, K)$. Then there is a reduction map

$$r : \mathcal{H} \longrightarrow \mathcal{T}$$

such that the preimage of a closed finite subtree is an affinoid space. Hence we can associate to every finite subtree $\mathcal{S}$ the K-Banach algebra $\mathcal{O}_{\mathcal{H}}\left(r^{-1}(\mathcal{S})\right)$. However this Banach algebra is a completion of a localization of $K[x]$ and the one variable polynomial ring $K[x]$ has no obvious quantization. Instead of trying to quantize the one-dimensional space $\mathcal{H}$, it is therefore more promising to try to quantize a higher-dimensional space related to $\mathcal{H}$. Such a space is given by the affine analogue $\mathcal{A}$ of $\mathcal{H}$, whose $\mathbb{C}_p$-valued points are just

$$\mathbb{A}^2(\mathbb{C}_p) - \{K\text{-rational hyperplanes containing } 0\}.$$

Let $\mathcal{N}_e$ be the set of valuations on K^2. Then there is a reduction map

$$r : \mathcal{A} \longrightarrow \mathcal{N}_e.$$

The tree $\mathcal{T}$ can be realized as the set of equivalence classes of valuations on K^2. Thus we have a canonical map $\mathrm{pr} : \mathcal{N}_e \to \mathcal{T}$. Let $\bar{r} = \mathrm{pr} \circ r$. The preimage under $\bar{r}$ of a finite subtree is an increasing union of affinoid spaces. Hence in this case we can associate to a finite subtree $\mathcal{S}$ the topological K-algebra $\mathcal{O}_{\mathcal{A}}\left(\bar{r}^{-1}(\mathcal{S})\right)$, which is the projective limit of K-Banach algebras.

The goal of this chapter is to provide a quantum analogue of this classical (nonquantum) construction. We will determine an infinite subtree $\mathcal{T}_q \subseteq \mathcal{T}$, and to every subtree $\mathcal{S} \subseteq \mathcal{T}_q$ we will associate a noncommutative topological K-algebra analogue $\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$ of $\mathcal{O}_{\mathcal{H}}\left(r^{-1}(\mathcal{S})\right)$ resp. $\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_e)$ of $\mathcal{O}_{\mathcal{A}}\left(\bar{r}^{-1}(\mathcal{S})\right)$, where $\mathcal{S}_e = \mathrm{pr}^{-1}(\mathcal{S})$. If $\mathcal{S} \subseteq \mathcal{T}_q$ is a finite subtree, the algebra $\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$ will be a K-Banach algebra. For two subtrees $\mathcal{S}_1 \subseteq \mathcal{S}_2$ of $\mathcal{T}_q$ we will obtain morphisms

$$\mathcal{O}_{\mathcal{T}_q}(\mathcal{S}_2) \longrightarrow \mathcal{O}_{\mathcal{T}_q}(\mathcal{S}_1)$$
$$\mathcal{O}_{\mathcal{N}_{q,e}}\left((\mathcal{S}_2)_e\right) \longrightarrow \mathcal{O}_{\mathcal{N}_{q,e}}\left((\mathcal{S}_1)_e\right).$$

The assignments $\mathcal{S} \mapsto \mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$ resp. $\mathcal{S} \mapsto \mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_e)$ define presheafs on the category of subtrees of $\mathcal{T}_q$, which are our quantum analogues of $\mathcal{H}$ resp. $\mathcal{A}$.

In order to motivate our construction in the quantized case we will first describe the picture in the commutative case. Here we will only sketch the constructions and won't give any proofs.
Let $v \in \mathcal{V}(\mathcal{T})$ be a vertex of $\mathcal{T}$. We will now provide a partial description of the Berkovich-analytic space $\mathcal{A}$, by describing the subsets $r^{-1}(v)$ as unions $r^{-1}(v) = \bigcup_n \mathcal{A}_{v,n}$, and by describing the algebra $\mathcal{O}_{\mathcal{A}}(\mathcal{A}_{v,n})$ and their completion with respect to the spectral seminorm.
By [DT08] Lemma 1.3.7 the set of $\mathbb{C}_p$-points of $r^{-1}(v) \subseteq \mathcal{H}$ is isomorphic to

$$\left\{[x,1] \in \mathbb{P}^1(\mathbb{C}_p) : \nu(x-t) = 0 \text{ for all } t \in \mathcal{O}\right\}.$$

Since $\mathcal{H}$ is a projectivization of $\mathcal{A}$ we can conclude that the set of $\mathbb{C}_p$-points of $\bar{r}^{-1}(v) \subseteq \mathcal{A}$ is isomorphic to

$$W := \left\{(x,y) \in \mathbb{C}_p^2 : \nu(ax+by) = \min\{\nu(x),\nu(y)\} \text{ for all } (a,b) \in \mathcal{O}^2 - \pi\mathcal{O}^2\right\}.$$

For $n \in \mathbb{N}_0$ let

$$\mathcal{A}_{v,n} := \{(x,y) \in W : -n \leq \min\{\nu(x),\nu(y)\} \leq \max\{\nu(x),\nu(y)\} \leq n\}.$$

The $\mathbb{C}_p$-points of $\bar{r}^{-1}(v)$ can be described as $\bigcup_n \mathcal{A}_{v,n}$. The affinoid algebra attached to $\mathcal{A}_{v,n}$ is given by

$$\mathcal{O}_{\mathcal{A}}\left(\mathcal{A}_{v,n}\right) = K\left\langle \pi^n X, \pi^n Y, \frac{\pi^n}{X}, \frac{\pi^n}{Y}, \frac{aX+bY}{cX+dY} : (a,b),(c,d) \in \mathcal{O}^2 - \pi\mathcal{O}^2 \right\rangle \tag{4.1.1}$$

and $\mathcal{O}_{\mathcal{A}}\left(\bar{r}^{-1}(v)\right) = \varprojlim_n \mathcal{O}_{\mathcal{A}}\left(\mathcal{A}_{v,n}\right)$.
Using the description of $\mathcal{O}_{\mathcal{A}}\left(\mathcal{A}_{v,n}\right)$ in (4.1.1) we see that there is a canonical K-algebra morphism $K[X,Y] \to \mathcal{O}_{\mathcal{A}}\left(\bar{r}^{-1}(v)\right)$. Thus a Berkovich point $\mathfrak{a}$ of $\bar{r}^{-1}(v)$ induces a multiplicative semivaluation $\mathfrak{a}$ on $K[X,Y]$. One can show that the Berkovich points of $\mathcal{O}_{\mathcal{A}}\left(\mathcal{A}_{v,n}\right)$ are exactly the Berkovich points $\mathfrak{a}$ of $\bar{r}^{-1}(v)$ that restrict to a multiplicative semivaluation on $K[X,Y]$ with $\mathfrak{a}\big(\mathcal{O}X + \mathcal{O}Y - \pi(\mathcal{O}X + \mathcal{O}Y)\big) \subseteq [-n,n]$.
The Berkovich points of $\mathcal{O}_{\mathcal{A}}\left(\mathcal{A}_{v,n}\right)$ admit another description, which we will now recall. In chapter 4.3 we will show that a subtree $\mathcal{S} \subseteq \mathcal{T}$ can be characterized by a set $A(\mathcal{S}) \subseteq K^2 \times K^2 \times [0,1)$ such that

$$[\alpha] \in \mathcal{S} \Leftrightarrow \alpha(f) + r \geq \alpha(g) \text{ for all } (f,g,r) \in A(\mathcal{S}).$$

If we identify K^2 with $KX \oplus KY \subseteq K[X,Y]$, we can view $A(\{v\})$ as a subset of $K[X,Y] \times K[X,Y] \times [0,1)$. Let $M(K[X,Y])$ be the set of multiplicative semivaluations on $K[X,Y]$ and let $H_1 := \mathcal{O}X + \mathcal{O}Y - \pi(\mathcal{O}X + \mathcal{O}Y)$. Then the Berkovich points of $\mathcal{O}_{\mathcal{A}}\left(\mathcal{A}_{v,n}\right)$ are in one to one correspondence with the set

$$B(v,n) := \left\{ \begin{array}{cc} \mathfrak{a} \in M(K[X,Y]) : & \mathfrak{a}(f) + r \geq \mathfrak{a}(g) \text{ for all } (f,g,r) \in A(v) \\ & \text{and } \mathfrak{a}(H_1) \subseteq [-n,n] \end{array} \right\}.$$

Let T be the multiplicative subset of $K[X,Y]$ generated by H_1. Then every $\mathfrak{a} \in B(v,n)$ extends to a multiplicative semivaluation on $T^{-1}K[X,Y]$ and we define the valuation $\nu_{v,n}$ on $T^{-1}K[X,Y]$ by

$$\nu_{v,n}\left(s^{-1}f\right) = \min\left\{\mathfrak{a}(f) - \mathfrak{a}(s) : \mathfrak{a} \in B(v,n)\right\}$$

for $s \in T, f \in K[X,Y]$. Then the completion of $\mathcal{O}_{\mathcal{A}}\left(\mathcal{A}_{v,n}\right)$ with respect to the spectral semivaluation is the same as the completion of $T^{-1}K[X,Y]$ with respect to $\nu_{v,n}$. For the spectral semivaluation and its relation to the Berkovich spectrum see e.g. [Ber90] chapter 1.3. Here we only discussed the case of a vertex $v \in \mathcal{T}$, but there is a very similar story for finite subtrees $\mathcal{S} \subset \mathcal{T}$.

We will now describe our construction in the quantized case. The problem that we have to overcome is that in general there are very few K-rational points on noncommutative rings. As a substitute we will use the Bruhat-Tits tree and quasi abelian multiplicative semivaluations, which are part of the

Berkovich spectrum.
Instead of using the algebra $K[x,y]$ of functions on the classical affine plane, we will work with the quantum plane $K[x,y]_q := K\{x,y\}/(xy - qyx)$. In section 4.2 we will construct a certain class of quasi abelian multiplicative valuations on $K[x,y]_q$. We will relate this class of valuations to a subset of $\mathcal{T}$ in section 4.4. In section 4.3 we will review some features of the Bruhat-Tits tree and show that a subtree $\mathcal{S}$ can be described by a set $A(\mathcal{S})$ as above.
In section 4.4 we will attach a set $U_{r_1,r_2}(\mathcal{S})$ of quasi abelian multiplicative semivaluations on $K[x,y]_q$ to a finite subtree $\mathcal{S}$ and $r_1 > r_2$, using the set $A(\mathcal{S})$. The set $U_{r_1,r_2}(\mathcal{S})$ will serve as an analogue of $\mathcal{A}_{v,n}$.
Let T be the multiplicative subset of $K[x,y]_q$ generated by

$$H_1 := \mathcal{O}x + \mathcal{O}y - \pi(\mathcal{O}x + \mathcal{O}y).$$

Similarly as in the commutative case, we would like to complete the localization of $K[x,y]_q$ at T by $\nu_{\mathcal{S},n}$ which should be defined as

$$\nu_{\mathcal{S},n}\left(s^{-1}f\right) := \min\{\mathfrak{a}(f) - \mathfrak{a}(s) : \mathfrak{a} \in U_{n,-n}(\mathcal{S})\}$$

for $s \in T, f \in K[x,y]_q$. This completion could then serve as an analogue of the completion of $\mathcal{O}_{\mathcal{A}}(\mathcal{A}(v,n))$ with respect to the spectral seminorm.
The problem is that localization is in general badly behaved in the noncommutative setting and that $\nu_{\mathcal{S},n}$ is in general not well defined. We will solve this problem by using the theory of algebraic microlocalization of P. Schneider, see [Záb12].

A multiplicative semivaluation ν on a K-algebra A is called quasi abelian if there exists $\gamma > 0$ such that for all $a, b \in A$ we have that $\nu(ab-ba) \geq \gamma+\nu(ab)$. With the technique of algebraic microlocalization one can construct for a K-algebra A, a multiplicative set $S \subset A$ and a finite set of multiplicative quasi abelian valuations M a K-Banach algebra $(A\langle M,S\rangle, \nu_M)$ together with a K-algebra morphism $\eta : A \to A\langle M,S\rangle$ such that $\eta(s) \in A\langle M,S\rangle^\times$ for $s \in S$. Unfortunately we cannot directly apply this construction to our case because the set $U_{r_1,r_2}(\mathcal{S})$ is infinite and may contain semivaluations with nontrivial kernels.
This problem is solved in section 4.4. There we first will define a subtree $\mathcal{T}_q$ and for $\mathcal{S} \subseteq \mathcal{T}_q$ a map $\iota : \mathcal{S}_{r_1,r_2} \longrightarrow U_{r_1,r_2}(\mathcal{S})$, which will be a section of the reduction map that we will construct. Here $\mathcal{S}_{r_1,r_2}$ is a subset of $\mathrm{pr}^{-1}(\mathcal{S})$ depending on r_1, r_2 where $\mathrm{pr} : \mathcal{N}_e \to \mathcal{T}$ is the map sending a valuation to its equivalence class. The image of ι only contains quasi abelian multiplicative valuations as described in section 4.2, which we know explicitly.
We will show in section 4.4 that it is possible to single out a finite subset

$\Lambda_{\mathcal{S}_{r_1,r_2}} \subseteq \iota\left(\mathcal{S}_{r_1,r_2}\right)$ such that for all $f \in K[x,y]_q$, $s \in T$ and $\mathfrak{a} \in U_{r_1,r_2}(\mathcal{S})$ there exists an $\alpha \in \Lambda_{\mathcal{S}_{r_1,r_2}}$ such that

$$\alpha(f) - \alpha(s) \leq \mathfrak{a}(f) - \mathfrak{a}(s).$$

Thus on a hypothetical localization of $K[x,y]_q$ by T we would have

$$\nu_{\mathcal{S},n}\left(s^{-1}f\right) = \min\{\alpha(f) - \alpha(s) : \alpha \in \Lambda_{\mathcal{S}_{r_1,r_2}}\}. \tag{4.1.2}$$

for $s \in T$, $f \in K[x,y]_q$. Hence we have overcome the problem that $U_{r_1,r_2}(\mathcal{S})$ is infinite and may contain semivaluations with nontrivial kernels.
Since $\Lambda_{\mathcal{S}_{r_1,r_2}}$ is a finite set consisting of quasi abelian multiplicative valuations we can attach to a finite subtree $\mathcal{S} \subseteq \mathcal{T}_q$ and $r_1 \geq r_2$ the K-Banach algebra

$$\mathcal{O}_{\mathcal{N}_{q,e}}\left(\mathcal{S}_{r_1,r_2}\right) := K[x,y]_q \left\langle \Lambda_{\mathcal{S}_{r_1,r_2}}, T \right\rangle.$$

For a K-Banach algebra A, let $\operatorname{Spmqb}(A)$ be the set of continuous quasi abelian multiplicative semivaluations. Then the observation (4.1.2) will enable us to show that $\operatorname{Spmqb}\left(\mathcal{O}_{\mathcal{N}_{q,e}}\left(\mathcal{S}_{r_1,r_2}\right)\right) \cong U_{r_1,r_2}(\mathcal{S})$ and hence we obtain a reduction map

$$\operatorname{Spmqb}\left(\mathcal{O}\left(\mathcal{S}_{r_1,r_2}\right)\right) \cong U_{r_1,r_2}(\mathcal{S}) \longrightarrow \mathcal{S}_{r_1,r_2}$$

by restricting elements $\mathfrak{a} \in U_{r_1,r_2}$ to $Kx \oplus Ky \cong K^2$.
For a possibly infinite subtree $\mathcal{S} \subseteq \mathcal{T}_q$ we will define locally convex K-algebras $\mathcal{O}_{\mathcal{N}_{q,e}}\left(\mathcal{S}_{r_1,r_2}\right)$ which are projective limits of K-Banach algebras associated to finite subtrees.
In section 4.6 we define a quantum analogue of $\mathcal{H}$. For a subtree $\mathcal{S} \subseteq \mathcal{T}_q$ we will define a subalgebra $\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$ of $\mathcal{O}_{\mathcal{N}_{q,e}}\left(\mathcal{S}_e\right)$ which basically consists of sums

$$\sum s_i^{-1} f_i \in \mathcal{O}_{\mathcal{N}_{q,e}}\left(\mathcal{S}_e\right)$$

with $s_i \in T$, $f_i \in K[x,y]_q$ and $\deg(s_i) = \deg(f_i)$. For a finite subtree $\mathcal{S} \subset \mathcal{T}_q$ the algebra $\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$ is a K-Banach algebra and we obtain a reduction map

$$r : \operatorname{Spmqb}\left(\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})\right) \longrightarrow \mathcal{S}.$$

The assignment $\mathcal{S} \mapsto \mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$ defines a presheaf on the subtrees of $\mathcal{T}_q$ and it is our quantized p-adic upper half plane.

4.2 Quasi abelian valuations on $K[x,y]_q$

In this section we will define and analyze a certain class of quasi abelian multiplicative valuations on $K[x,y]_q$.

Definition 4.2.1. Let R be a K-algebra. A multiplicative quasi abelian semi-valuation $\gamma : R \longrightarrow \mathbb{R} \cup \{\infty\}$ on R of level $r > 0$ is a multiplicative semivaluation with the property that

$$\gamma(ab - ba) \geq \gamma(ab) + r$$

for all $a, b \in R$. A multiplicative semivaluation γ on R is called quasi abelian if there exists an $r > 0$ such that γ is a multiplicative quasi abelian semi-valuation of level r. We denote the set of multiplicative quasi abelian semi-valuations on R by $\mathrm{Spmq}(R)$.

Definition 4.2.2. For $g = \begin{pmatrix} g_a & g_b \\ g_c & g_d \end{pmatrix} \in M(2,K)$ and $x, y \in K\{x,y\}$ we define $gx := g_a x + g_c y$ and $gy := g_b x + g_d y$. In this way an element $g \in M(2,K)$ defines an endomorphism of $K\{x,y\}$. Because of

$$g(hx) = g(h_a x + h_c y) = (g_a h_a + g_b h_c)x + (g_c h_a + g_d h_c)y = (gh)_a x + (gh)_c y$$
$$g(hy) = g(h_b x + h_d y) = (g_a h_b + g_b h_d)x + (g_c h_b + g_d h_d)y = (gh)_b x + (gh)_d y$$

this defines a left action of $\mathrm{GL}(2,K)$ on $K\{x,y\}$.

Definition 4.2.3. Let $G(q) \subseteq \mathrm{GL}(2,K)$ be the set of matrices of the form $g = \begin{pmatrix} g_a & g_b \\ g_c & g_d \end{pmatrix}$ fulfilling

$$\nu(\det g) < \nu(1-q) + \min\{\nu(g_a), \nu(g_c)\} + \min\{\nu(g_b), \nu(g_d)\}.$$

For an element $g \in \mathrm{GL}(2,K)$ we define

$$\tau_g = \nu(1-q) + \min\{\nu(g_a), \nu(g_c)\} + \min\{\nu(g_b), \nu(g_d)\} - \nu(\det g).$$

In particular $G(q) = \{g \in \mathrm{GL}(2,K) : \tau_g > 0\}$

Definition 4.2.4. Let

$$Z := \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in K^\times \right\}$$

and let

$$T := \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in K^\times \right\}.$$

Lemma 4.2.5. *Let $h \in G(q)$.*

1. *Let $g \in \mathrm{GL}(2, \mathcal{O})$. Then $\tau_h = \tau_{gh}$ and thus $\mathrm{GL}(2, \mathcal{O})G(q) = G(q)$.*

2. *Let $g \in Z \subset \mathrm{GL}(2, K)$. Then we have that $\tau_{gh} = \tau_{hg} = \tau_h$ and thus $ZG(q) = G(q)Z = G(q)$.*

3. *Let $t \in T$. Then $\tau_h = \tau_{ht}$ and and thus $G(q)T = G(q)$.*

Proof. 1. Let $g \in \mathrm{GL}(2, \mathcal{O})$, $(w, z)^t \in K^2$ and $(\alpha, \beta) = g(w, z)^t$. Then

$$\min\{\nu(\alpha), \nu(\beta)\} = \min\{\nu(w), \nu(z)\}.$$

Thus for $h = \begin{pmatrix} h_a & h_b \\ h_c & h_d \end{pmatrix} \in G(q)$

$$\begin{aligned} \min\{\nu((gh)_a), \nu((gh)_c)\} &= \min\{\nu(h_a), \nu(h_c)\} \\ \min\{\nu((gh)_b), \nu((gh)_d)\} &= \min\{\nu(h_b), \nu(h_d)\}. \end{aligned}$$

Since $\nu(\det h) = \nu(\det gh)$ we can conclude $\tau_h = \tau_{gh}$.
2. This is immediate.
3. Since $\begin{pmatrix} h_a & h_b \\ h_c & h_d \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} = \begin{pmatrix} zh_a & wh_b \\ zh_c & wh_d \end{pmatrix}$ this follows from the definition of τ_g. □

Lemma 4.2.6. *Let $g \in G(q)$ and let $I \subseteq K\{x, y\}$ be the two sided ideal generated by $gxgy - qgygx$.*
Then there exists a homogeneous polynomial $f_g \in \mathcal{O}\{x, y\}$ of degree two such that

$$I = (xy - yx + \pi^{\tau_g} f_g).$$

Proof. Recall that we defined $\det_q(g) := g_a g_d - q g_b g_c$. For $g \in G(q)$ we know that

$$\begin{aligned} \nu(\det g) &< \nu(1 - q) + \min\{\nu(g_a), \nu(g_c)\} + \min\{\nu(g_b), \nu(g_d)\} \\ &\leq \nu(1 - q) + \nu(g_c) + \nu(g_b) \\ &= \nu((1 - q) g_b g_c) \end{aligned}$$

and thus

$$\begin{aligned} \nu(\det_q(g)) &= \nu(g_a g_d - q g_b g_c) \\ &= \nu(g_a g_d - g_b g_c + (1 - q) g_b g_c) \\ &= \nu(g_a g_d - g_b g_c) \\ &= \nu(\det g) \neq \infty. \end{aligned} \tag{4.2.1}$$

Furthermore

$$
\begin{aligned}
&gxgy - qgygx \\
&= (g_a x + g_c y)(g_b x + g_d y) - q(g_b x + g_d y)(g_a x + g_c y) \\
&= xy(g_a g_d - q g_b g_c) + yx(g_b g_c - q g_a g_d) + x^2 g_a g_b (1-q) + y^2 g_c g_d (1-q)
\end{aligned}
$$

Since $\det_q(g) = g_a g_d - q g_b g_c \neq 0$ the ideal I is also generated by

$$
xy - yx + yx\left(\frac{g_b g_c - q g_a g_d}{\det_q(g)} + 1\right) + x^2 \frac{g_a g_b(1-q)}{\det_q(g)} + y^2 \frac{g_c g_d (1-q)}{\det_q(g)}.
$$

and thus it is enough to show that the valuations of the three last coefficients are at least τ_g. For the first one consider

$$
\frac{g_b g_c - q g_a g_d}{\det_q(g)} + 1 = \frac{g_b g_c - q g_a g_d}{g_a g_d - q g_b g_c} + 1 = \frac{(1-q) g_b g_c}{\det_q(g)} + \frac{(1-q) g_a g_d}{\det_q(g)}.
$$

Since we already showed $\nu(\det_q(g)) = \nu(\det g)$ and since

$$
\begin{aligned}
\nu(g_b g_c) &\geq \min\{\nu(g_a), \nu(g_c)\} + \min\{\nu(g_b), \nu(g_d)\} \\
\nu(g_a g_d) &\geq \min\{\nu(g_a), \nu(g_c)\} + \min\{\nu(g_b), \nu(g_d)\}
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\nu\left(\frac{g_b g_c - q g_a g_d}{\det_q(g)} + 1\right) &= \nu\left(\frac{(1-q) g_b g_c}{\det_q(g)} + \frac{(1-q) g_a g_d}{\det_q(g)}\right) \\
&\geq \nu(1-q) + \min\{\nu(g_a), \nu(g_c)\} + \min\{\nu(g_b), \nu(g_d)\} \\
&\quad - \nu(\det(g)) \\
&= \tau_g.
\end{aligned}
$$

The estimates of the coefficients of x^2 and y^2 can be obtained similarly. $\square$

Definition 4.2.7. For a monomial $m \in K\{x, y\}$ we will denote by $\deg_x m$ the number of times that x occurs in m and by $\deg_y m$ the number of times that y occurs in m. Moreover we define $\deg m := \deg_x m + \deg_y m$. For $\mu \in \mathbb{R}^2$ we denote by ν_μ the multiplicative valuation on $K\{x, y\}$ which is defined on a monomial $m \in K\{x, y\}$ as $\nu_\mu(m) := \mu_1 \deg_x(m) + \mu_2 \deg_y(m)$. For a finite set of monomials U, $a_m \in K$ and $f = \sum_{m \in U} a_m m$ it is given by

$$
\nu_\mu(f) := \inf\{\nu(a_m) + \nu_\mu(m) : m \in U\}.
$$

The multiplicativity is shown in the same way as one shows that the Gauß valuation on a Tate algebra is multplicative, see e.g. [BGR84] section 5.1.2 Proposition 1. We denote the completion of $K\{x, y\}$ with respect to ν_μ by $K_\mu\langle x, y\rangle$.

Definition 4.2.8. For $g \in \mathrm{GL}(2, K)$ and $\mu \in \mathbb{R}^2$ we define

$$\epsilon_{g,\mu} := \tau_g - |\mu_1 - \mu_2|.$$

Lemma 4.2.9. *Let $g \in G(q)$ and let $\mu \in \mathbb{R}^2$ be such that $\epsilon_{g,\mu} > 0$. Moreover let $\iota_g := xy - yx + \pi^{\tau_g} f_g$ where f_g is as in Lemma 4.2.6. Let $m \in K\{x, y\}$ be a monomial with $\deg_x m = s$ and $\deg_y m = t$. Then there exist $b_{m_1,m_2} \in \mathcal{O}$ and $a_j \in \mathcal{O}$ such that in $K\{x, y\}$ we have*

$$m = x^s y^t + \sum_{j+k=s+t} a_j x^j y^k + \sum_{\deg m_1 + \deg m_2 = \deg m - 2} b_{m_1,m_2} m_1 \iota_g m_2$$

and $\nu_\mu\left(a_j x^j y^k\right) \geq \nu_\mu\left(x^s y^t\right) + \epsilon_{g,\mu}$ and $\nu_\mu\left(b_{m_1,m_2} m_1 \iota_g m_2\right) \geq \nu_\mu(m)$.

Proof. We have that

$$yx = xy - \iota_g + \pi^{\tau_g} f_g. \tag{4.2.2}$$

Because $f_g \in \mathcal{O}\{x, y\}$ is a homogeneous polynomial of degree 2 we know that $\nu_\mu(\pi^{\tau_g} f_g) \geq \nu_\mu(xy) + \epsilon_{g,\mu}$. Let n_1, n_2 be monomials. Then the multiplicativity of ν_μ ensures that we have

$$\nu_\mu\left(n_1 \pi^{\tau_g} f_g n_2\right) \geq \nu_\mu\left(n_1 yx n_2\right) + \epsilon_{g,\mu} = \nu_\mu\left(n_1 xy n_2\right) + \epsilon_{g,\mu} \tag{4.2.3}$$

and thus

$$\nu_\mu\left(n_1 \iota_g n_2\right) \geq \nu_\mu\left(n_1 yx n_2\right). \tag{4.2.4}$$

Equation (4.2.2) and the estimates (4.2.3), (4.2.4) imply that there exist elements $b^{(0)}_{m_1,m_2} \in \mathcal{O}$ and $c^{(0)}_{m'} \in \mathcal{O}$ such that

$$m = x^s y^t - \sum_{\deg m_1 + \deg m_2 = \deg m - 2} b^{(0)}_{m_1,m_2} m_1 \iota_g m_2 + \sum_{\deg m' = \deg m} c^{(0)}_{m'} m'$$

with $\nu_\mu(b^{(0)}_{m_1,m_2} m_1 \iota_g m_2) \geq \nu_\mu(m)$ and $\nu_\mu(c^{(0)}_{m'} m') \geq \nu_\mu(m) + \epsilon_{g,\mu}$.
Let $\tilde{m}$ be a monomial with $\deg \tilde{m} = \deg m$. Now we can use the same strategy as above to find $b^{\tilde{m}}_{m_1,m_2} \in \mathcal{O}$ and $c^{\tilde{m}}_{m'} \in \mathcal{O}$ such that

$$\begin{aligned} c^{(0)}_{\tilde{m}} \tilde{m} = c^{(0)}_{\tilde{m}} x^{\deg_x \tilde{m}} y^{\det_y \tilde{m}} - \sum_{\deg m_1 + \deg m_2 = \deg m - 2} b^{\tilde{m}}_{m_1,m_2} m_1 \iota_g m_2 \\ + \sum_{\deg m' = \deg \tilde{m}} c^{\tilde{m}}_{m'} m' \end{aligned}$$

with

$$\begin{aligned}\nu_\mu\left(c^{(0)}_{\tilde{m}} x^{\deg_x \tilde{m}} y^{\det_y \tilde{m}}\right) &= \nu_\mu\left(c^{(0)}_{\tilde{m}} \tilde{m}\right) \geq \nu_\mu(m) + \epsilon_{g,\mu}\\ \nu_\mu\left(b^{\tilde{m}}_{m_1,m_2} m_1 \iota_g m_2\right) &\geq \nu_\mu\left(c^{(0)}_{\tilde{m}} \tilde{m}\right) \geq \nu_\mu(m) + \epsilon_{g,\mu}\\ \nu_\mu\left(c^{\tilde{m}}_{m'} m'\right) &\geq \nu_\mu\left(c^{(0)}_{\tilde{m}} \tilde{m}\right) + \epsilon_{g,\mu} \geq \nu_\mu(m) + 2\epsilon_{g,\mu}.\end{aligned}$$

Thus for

$$a^{(1)}_j = \sum_{\substack{\deg m' = \deg m\\ \deg_x m' = j}} c^{(0)}_{m'}; \qquad b^{(1)}_{m_1,m_2} = \sum_{\deg \tilde{m} = \deg m} b^{\tilde{m}}_{m_1,m_2}$$

$$c^{(1)}_{m'} = \sum_{\deg \tilde{m} = \deg m'} c^{\tilde{m}}_{m'}$$

we know that

$$\begin{aligned}m =& x^s y^t + \sum_{j+k=s+t} a^{(1)}_j x^j y^k + \sum_{\deg m' = \deg m} c^{(1)}_{m'} m'\\ &+ \sum_{\deg m_1 + \deg m_2 = \deg m - 2} \left(\sum_{r=0}^{1} b^{(r)}_{m_1,m_2}\right) m_1 \iota_g m_2.\end{aligned}$$

and

$$\begin{aligned}\nu_\mu\left(a^{(1)}_j x^j y^k\right) &\geq \nu_\mu(m) + \epsilon_{g,\mu}\\ \nu_\mu\left(b^{(1)}_{m_1,m_2} m_1 \iota_g m_2\right) &\geq \nu_\mu(m) + \epsilon_{g,\mu}\\ \nu_\mu\left(c^{(1)}_{m'} m'\right) &\geq \nu_\mu(m) + 2\epsilon_{g,\mu}.\end{aligned}$$

Repeating this process we can find for $r \in \mathbb{N}_0$ the following.

1. $a^{(r)}_j \in \mathcal{O}$ with $\nu_\mu\left(a^{(r)}_j x^j y^k\right) \geq \nu_\mu(m) + r\epsilon_{g,\mu}$.
2. $b^{(r)}_{m_1,m_2} \in \mathcal{O}$ with $\nu_\mu(b^{(r)}_{m_1,m_2} m_1 \iota_g m_2) \geq \nu_\mu(m) + r\epsilon_{g,\mu}$.
3. $c^{(r)}_{m'} \in \mathcal{O}$ with $\nu_\mu(c^{(r)}_{m'} m') \geq \nu_\mu(m) + (r+1)\epsilon_{g,\mu}$

such that for $n \in \mathbb{N}_0$

$$\begin{aligned}m =& x^s y^t + \sum_{j+k=s+t} \left(\sum_{r=1}^{n} a^{(r)}_j\right) x^j y^k + \sum_{\deg m' = \deg m} c^{(n)}_{m'} m'\\ &+ \sum_{\deg m_1 + \deg m_2 = \deg m - 2} \left(\sum_{r=0}^{n} b^{(r)}_{m_1,m_2}\right) m_1 \iota_g m_2.\end{aligned}$$

The estimates in 1., 2. and 3. imply that

$$a_j := \sum_{r\geq 1} a_j^{(r)} \quad \text{and} \quad b_{m_1,m_2} := \sum_{r\geq 0} b_{m_1,m_2}^{(r)}$$

exist and that $\lim_{n\to\infty} c_{m'}^{(n)} m' = 0$. Thus

$$m = x^s y^t + \sum_{j+k=s+t} a_j x^j y^k + \sum_{\deg m_1 + \deg m_2 = \deg m - 2} b_{m_1,m_2} m_1 \iota_g m_2.$$

The inequalities in the Lemma follow from the inequalities in 1., 2., 3. $\square$

Lemma 4.2.10. *Let $g^{-1} \in G(q)$ and let $I_{g^{-1}} = \big(g^{-1}xg^{-1}y - qg^{-1}yg^{-1}x\big)$. Then the set $\{x^i y^j : i,j \in \mathbb{N}_0\}$ is a K-basis of $K\{x,y\}/I_{g^{-1}}$ and the set $\{(gx)^i(gy)^j : i,j \in \mathbb{N}_0\}$ is a K-basis of $K[x,y]_q$.*

Proof. By Lemma 4.2.6 the two sided ideal $I_{g^{-1}}$ is generated by

$$\iota_{g^{-1}} = xy - yx + \pi^{\tau_{g^{-1}}} f_{g^{-1}}$$

for $f_{g^{-1}}$ as described in Lemma 4.2.6. Thus $K\{x,y\}/I_{g^{-1}} = K\{x,y\}/(\iota_{g^{-1}})$ and we will show the statement for $K\{x,y\}/(\iota_{g^{-1}})$.
Since $xy - qyx$ is in the kernel of

$$\begin{array}{ccc} K\{x,y\} & \longrightarrow & K\{x,y\}/(\iota_{g^{-1}}) \\ x & \longmapsto & g^{-1}x \\ y & \longmapsto & g^{-1}y \end{array}$$

we obtain an algebra morphism $K[x,y]_q \xrightarrow{\phi_{g^{-1}}} K\{x,y\}/(\iota_{g^{-1}})$. Similarly we obtain an algebra morphism $K\{x,y\}/(\iota_{g^{-1}}) \xrightarrow{\phi_g} K[x,y]_q$ by sending x resp. y to gx resp. gy. Then $\phi_g\phi_{g^{-1}} = \mathrm{id}$ and $\phi_{g^{-1}}\phi_g = \mathrm{id}$. Thus both maps are isomorphisms.

We will first show that $\{x^i y^j : i,j \in \mathbb{N}_0\}$ is a K-basis of $K\{x,y\}/(\iota_{g^{-1}})$. Note that $K\{x,y\}$ is a graded ring with n-th graded part being the K-vector space of homogeneous polynomials of degree n. Since $(\iota_{g^{-1}})$ is a graded ideal, also $K\{x,y\}/(\iota_{g^{-1}})$ is a graded ring with n-th graded part

$$A_n := \{f + (\iota_{g^{-1}}) : f \in K\{x,y\} \text{ is a homogeneous polynomial of degree } n\}$$

and A_n is a finite dimensional K-vector space. We know that

$$\phi_{g^{-1}}(x^i y^j) = (g^{-1}x)^i (g^{-1}y)^j \in A_n$$

for $i+j=n$. Since $\phi_{g^{-1}}$ is an isomorphism the set

$$\{\phi_{g^{-1}}(x^iy^j) : i+j=n\} = \{(g^{-1}x)^i(g^{-1}y)^j : i+j=n\} \subseteq A_n$$

is K-linearly independent since $\{x^iy^j : i,j \in \mathbb{N}_0\}$ is a K-basis of $K[x,y]_q$ (see e.g. [Kas95] Proposition IV.1.1.). This means that

$$\dim_K A_n \geq \#\{(i,j) \in \mathbb{N}_0^2 : i+j=n\}.$$

Note that for $g^{-1} \in G(q)$ and $\mu = (0,0)$ we have that $\epsilon_{g^{-1},\mu} > 0$. Thus by Lemma 4.2.9 A_n is generated as a K-vector space by $\{x^iy^j : i+j=n\}$. Since

$$\#\{x^iy^j : i+j=n\} = \#\{(i,j) \in \mathbb{N}_0^2 : i+j=n\} \leq \dim_K A_n.$$

we can conclude that $\{x^iy^j : i+j=n\}$ is a K-basis of A_n. Thus

$$K\{x,y\}/(\iota_{g^{-1}}) = \bigoplus_{n\in\mathbb{N}_0} A_n$$

implies that $\{x^iy^j : i,j \in \mathbb{N}_0\}$ is a K-basis of $K\{x,y\}/(\iota_{g^{-1}})$.
Because of

$$\{(gx)^i(gy)^j \in K[x,y]_q : i,j \in \mathbb{N}_0\} = \{\phi_g(x^iy^j) : i,j \in \mathbb{N}_0\}$$

and because ϕ_g is an isomorphism it is clear that $\{(gx)^i(gy)^j : i,j \in \mathbb{N}_0\}$ is a K-basis of $K[x,y]_q$. $\square$

Proposition 4.2.11. *Let $g \in G(q)$ and $\mu \in \mathbb{R}^2$ be such that $\epsilon_{g,\mu} > 0$. Let J_g be the closure of the two sided ideal $(gxgy - qgygx)$ in $K_\mu\langle x,y\rangle$ and let $\nu^{g,\mu}$ be the residue valuation on $K_\mu\langle x,y\rangle/J_g$. Then every element in $K_\mu\langle x,y\rangle/J_g$ can uniquely be written as a converging sum $\sum_{i,j} a_{i,j}x^iy^j$ with*

$$\nu^{g,\mu}\left(\sum_{i,j} a_{i,j}x^iy^j\right) = \inf\{\nu(a_{i,j}) + \mu_1 i + \mu_2 j\}.$$

Moreover $\nu^{g,\mu}$ is multiplicative and quasi abelian of level $\epsilon_{g,\mu}$

Proof. By Lemma 4.2.9 we can write every element of $K_\mu\langle x,y\rangle/J_g$ as a sum of the form $\sum_{i,j} a_{i,j}x^iy^j$ with $\lim \nu_\mu\left(a_{i,j}x^iy^j\right) = 0$.
Moreover

$$\nu^{g,\mu}\left(\sum_{i,j} a_{i,j}x^iy^j\right) \geq \nu_\mu\left(\sum_{i,j} a_{i,j}x^iy^j\right) = \inf\{\nu(a_{i,j}) + \mu_1 i + \mu_2 j\}.$$

Assume that $\nu^{g,\mu}\left(\sum_{i,j} a_{i,j}x^iy^j\right) > \inf\{\nu(a_{i,j}) + \mu_1 i + \mu_2 j\}$. Then there exists an $a \in J_g$ such that

$$\nu_\mu\left(\sum_{i,j} a_{i,j}x^iy^j + a\right) > \inf\{\nu(a_{i,j}) + \mu_1 i + \mu_2 j\}.$$

By the definition of ν_μ that means that there exist an $l \in \mathbb{N}_0$ and a homogeneous polynomial $w \in J_g$ of degree l and $b_m \in K$ such that

$$\sum_{i+j=l} a_{i,j}x^iy^j + w = \sum_{\deg m=l} b_m m$$

and $\nu_\mu\left(\sum_{\deg m=l} b_m m\right) > \nu_\mu\left(\sum_{i+j=l} a_{i,j}x^iy^j\right)$. By Lemma 4.2.9 we know that there exists a homogeneous polynomial $z \in J_g$ of degree l such that

$$\sum_{\deg m=l} b_m m = \sum_{i+j=l} c_{i,j}x^iy^j + z$$

with

$$\nu_\mu\left(\sum_{i+j=l} c_{i,j}x^iy^j\right) \geq \nu_\mu\left(\sum_{\deg m=l} b_m m\right) > \nu_\mu\left(\sum_{i+j=l} a_{i,j}x^iy^j\right).$$

Thus $\sum_{i+j=l}(a_{i,j} - c_{i,j})x^iy^j \neq 0$ and

$$\sum_{i+j=l}(a_{i,j} - c_{i,j})x^iy^j = z - w.$$

But since $z - w \in J_g \cap K\{x, y\} = I_g$ this contradicts Lemma 4.2.10. Thus

$$\nu\left(\sum_{i,j} a_{i,j}x^iy^j\right) = \inf\{\nu(a_{i,j}) + \mu_1 i + \mu_2 j\}.$$

This implies also that the presentation as a sum in x^iy^j is unique.
Let m_1, m_2 be monomials. By Lemma 4.2.9 and Lemma 4.2.6 and by the description of $\nu^{g,\mu}$ we just proved, we know that

$$\nu^{g,\mu}(m_1m_2) = \nu^{g,\mu}(m_1) + \nu^{g,\mu}(m_2)$$

and

$$\nu^{g,\mu}(m_1m_2 - m_2m_1) \geq \nu^{g,\mu}(m_1m_2) + \epsilon_{g,\mu}.$$

Thus the description of $\nu^{g,\mu}$ implies that $\nu^{g,\mu}$ is multiplicative and quasi abelian of level $\epsilon_{g,\mu}$. □

Definition 4.2.12. Let $g^{-1} \in G(q)$ and let $\mu \in \mathbb{R}^2$ such that $\epsilon_{g^{-1},\mu} > 0$. The

Lemmas 4.2.10 and 4.2.11 imply that the canonical map

$$K\{x,y\}/I_{g^{-1}} \to K_\mu\langle x,y\rangle/J_{g^{-1}}$$

is injective with dense image. Since $\phi_{g^{-1}} : K[x,y]_q \to K\{x,y\}/I_{g^{-1}}$ is an isomorphism we obtain an injective map

$$K[x,y]_q \longrightarrow K_\mu\langle x,y\rangle/J_{g^{-1}}$$

that we will again denote by $\phi_{g^{-1}}$. By $\nu_{g,\mu}$ we denote the quasi abelian multiplicative valuation of level $\epsilon_{g^{-1},\mu}$ on $K[x,y]_q$ defined by

$$\nu_{g,\mu} := \nu^{g^{-1},\mu} \circ \phi_{g^{-1}}.$$

We define $K_{g,\mu}\langle x,y\rangle := \widehat{K[x,y]_q}^{\nu_{g,\mu}}$.

Remark 4.2.13. Since $\phi_{g^{-1}} : K[x,y]_q \longrightarrow K_\mu\langle x,y\rangle/J_{g^{-1}}$ is injective with dense image, it extends to an isometry

$$K_{g,\mu}\langle x,y\rangle \longrightarrow K_\mu\langle x,y\rangle/J_{g^{-1}}.$$

Proposition 4.2.14. *Let $g^{-1} \in G(q)$ and let $\mu \in \mathbb{R}^2$ be such that $\epsilon_{g^{-1},\mu} > 0$. Every $f \in K_{g,\mu}\langle x,y\rangle$ can uniquely be written as a sum $\sum_{i,j} a_{i,j}(gx)^i(gy)^j$ with $\nu(a_{i,j}) + i\mu_1 + j\mu_2 \to \infty$. Moreover*

$$\nu_{g,\mu}\left(\sum_{i,j} a_{i,j}(gx)^i(gy)^j\right) = \min\{\nu(a_{i,j}) + i\mu_1 + j\mu_2\}.$$

Proof. By Lemma 4.2.10 we know that $\{(gx)^i(gy)^j : i,j \in \mathbb{N}_0\}$ is a basis of the dense subalgebra $K[x,y]_q \subseteq K_{g,\mu}\langle x,y\rangle$. Since for finite sums we know that

$$\begin{aligned}\nu_{g,\mu}\left(\sum_{i,j} a_{i,j}(gx)^i(gy)^j\right) &= \nu_\mu\left(\phi_{g^{-1}}\left(\sum_{i,j} a_{i,j}(gx)^i(gy)^j\right)\right)\\ &= \nu_\mu\left(\sum_{i,j} a_{i,j}x^iy^j\right)\\ &= \min\{\nu(a_{i,j}) + i\mu_1 + j\mu_2\}\end{aligned}$$

we can conclude the claim. $\square$

Lemma 4.2.15. *Let $g^{-1} \in G(q)$ and let μ be such that $\epsilon_{g^{-1},\mu} > 0$. Then there exist a $g'^{-1} \in G(q)$ and an $\eta \in \mathbb{R}^2$ with $|\eta_1 - \eta_2| < 1$ such that $\epsilon_{g'^{-1},\eta} > 0$ and $\nu_{g,\mu} = \nu_{g',\eta}$.*

Proof. We can assume that $\mu_2 \geq \mu_1$. Let $t = \begin{pmatrix} 1 & 0 \\ 0 & \pi^{\lfloor \mu_2 - \mu_1 \rfloor} \end{pmatrix}$. Because of

$$\begin{aligned}
\tau_{tg^{-1}} &= \nu(1-q) + \min\left\{\nu\left((g^{-1})_a\right), \nu\left((g^{-1})_c\right) + \lfloor \mu_2 - \mu_1 \rfloor\right\} \\
&\quad + \min\left\{\nu\left((g^{-1})_b\right), \nu\left((g^{-1})_d\right) + \lfloor \mu_1 - \mu_2 \rfloor\right\} - \nu\left(\det\left(g^{-1}\right)\right) - \lfloor \mu_2 - \mu_1 \rfloor \\
&\geq \nu(1-q) + \min\left\{\nu\left((g^{-1})_a\right), \nu\left((g^{-1})_c\right)\right\} \\
&\quad + \min\left\{\nu\left((g^{-1})_b\right), \nu\left((g^{-1})_d\right)\right\} - \nu\left(\det\left(g^{-1}\right)\right) - |\mu_2 - \mu_1| \\
&= \epsilon_{g^{-1},\mu} > 0
\end{aligned}$$

we know that $tg^{-1} \in G(q)$. For $g' = gt^{-1}$ and $\eta = (\mu_1, \mu_2 - \lfloor \mu_2 - \mu_1 \rfloor)$ have by definition

$$\epsilon_{g'^{-1},\eta} = \tau_{g'^{-1}} - |\eta_1 - \eta_2|.$$

Since $\tau_{g'^{-1}} \in \mathbb{Z} \cap \mathbb{R}_{>0}$ we know that $\tau_{g'^{-1}} \geq 1$. Thus

$$|\eta_1 - \eta_2| = |\mu_1 - (\mu_2 - \lfloor \mu_2 - \mu_1 \rfloor)| < 1$$

implies that $\epsilon_{g'^{-1},\eta} > 0$.
Let $f \in K[x,y]_q$. Then there exist $a_{i,j} \in K$ such that $f = \sum_{i,j} a_{i,j}(gx)^i(gy)^j$. Thus

$$\begin{aligned}
\nu_{g',\eta}(f) &= \nu^{g',\eta}(\phi_{g'^{-1}}(f)) \\
&= \nu^{g',\eta}\left(\sum_{i,j} a_{i,j}((tg^{-1}g)x)^i((tg^{-1}g)y)^j\right) \\
&= \nu^{g',\eta}\left(\sum_{i,j} a_{i,j}(tx)^i(ty)^j\right) \\
&= \nu^{g',\eta}\left(\sum_{i,j} a_{i,j}x^i\left(\pi^{\lfloor \mu_1 - \mu_2 \rfloor}y\right)^j\right) \\
&= \min\{\nu(a_{i,j}) + i\mu_1 + j(\lfloor \mu_2 - \mu_1 \rfloor + \mu_2 - \lfloor \mu_2 - \mu_1 \rfloor)\} \\
&= \min\{\nu(a_{i,j}) + i\mu_1 + j\mu_2\} \\
&= \nu^{g,\mu}(\phi_{g^{-1}}(f)) \\
&= \nu_{g,\mu}(f)
\end{aligned}$$

and hence $\nu_{g,\mu} = \nu_{g',\eta}$. $\square$

4.2.16. If $g^{-1} \in G(q)$ then $\tau_{g^{-1}} \geq 1$. Hence for $g^{-1} \in G(q)$ and $\mu \in \mathbb{R}^2$ with $|\mu_1 - \mu_2| < 1$ we have that $\epsilon_{g^{-1},\mu} > 0$. Thus Lemma 4.2.15 implies that

$$\left\{\nu_{g,\mu} : g^{-1} \in G(q) \text{ and } \epsilon_{g^{-1},\mu} > 0\right\} = \left\{\nu_{g,\mu} : g^{-1} \in G(q) \text{ and } |\mu_1 - \mu_2| < 1\right\}$$

and every element in this set is a quasi abelian and multiplicative valuation by the Lemmas 4.2.11 and 4.2.14.

Lemma 4.2.17. *Let $g^{-1} \in G(q)$, $h \in \mathrm{GL}(2,\mathcal{O})$ and let $\mu_1 = \mu_2$. Then also $(gh)^{-1} \in G(q)$ and $\nu_{gh,\mu} = \nu_{g,\mu}$.*

Proof. Lemma 4.2.5 implies that $(gh)^{-1} \in G(q)$ if both $h \in \mathrm{GL}(2,\mathcal{O})$ and $g^{-1} \in G(q)$. For $a \in G(q)$ let J_a closure of the two sided ideal generated by $axay - qayax$. To show $\nu_{gh,\mu} = \nu_{g,\mu}$ consider the map

$$K_\mu\langle x,y\rangle \xrightarrow{h^{-1}} K_\mu\langle x,y\rangle / J_{(gh)^{-1}}$$

If $\nu^{(gh)^{-1},\mu}(h^{-1}x) \geq \nu_\mu(x)$ and $\nu^{(gh)^{-1},\mu}(h^{-1}y) \geq \nu_\mu(y)$ then this map is well defined and valuation increasing. By definition

$$\nu^{(gh)^{-1},\mu}(x) = \nu_\mu(x) = \nu_\mu(y) = \nu^{(gh)^{-1},\mu}(y).$$

But since $h^{-1} \in \mathrm{GL}(2,\mathcal{O})$ this implies the previous inequalities and thus the map is well defined.
Since $J_{g^{-1}}$ is in the kernel we obtain a valuation increasing K-algebra morphism

$$\psi_{h^{-1}} : K_\mu\langle x,y\rangle / J_{g^{-1}} \longrightarrow K_\mu\langle x,y\rangle / J_{(gh)^{-1}}.$$

Applying the same argument for the map

$$K_\mu\langle x,y\rangle \xrightarrow{h} K_\mu\langle x,y\rangle / J_{(g)^{-1}},$$

where $J_{(gh)^{-1}}$ is in the kernel, we see that $\psi_{h^{-1}}$ is an isometry with inverse ψ_h. Thus the commutative diagram

$$\begin{tikzcd} K[x,y]_q \arrow[r, "\phi_{g^{-1}}"] \arrow[dr, "\phi_{(gh)^{-1}}"'] & K_\mu\langle x,y\rangle / J_{g^{-1}} \arrow[d, "\psi_{h^{-1}}"] \\ & K_\mu\langle x,y\rangle / J_{(gh)^{-1}} \end{tikzcd}$$

proves the claim. □

Definition 4.2.18. We denote by $B(\mathcal{O})$ the Iwahori subgroup of $\mathrm{GL}(2,\mathcal{O})$ given by

$$B(\mathcal{O}) = \left\{ \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \mathrm{GL}(2,\mathcal{O}) : \nu(c) \geq 1 \right\}.$$

Lemma 4.2.19. *Let $\mu_1 > \mu_2$ with $|\mu_1 - \mu_2| < 1$, $g^{-1} \in G(q)$ and let $h \in B(\mathcal{O})$. Then $(gh)^{-1} \in G(q)$ and $\nu_{gh,\mu} = \nu_{g,\mu}$.*

Proof. The first statement follows from Lemma 4.2.5. To show the equality $\nu_{gh,\mu} = \nu_{g,\mu}$ we use the same strategy as in Lemma 4.2.17.
We have that $\nu^{(gh)^{-1},\mu}(x) = \nu_\mu(x) = \mu_1$ and $\nu^{(gh)^{-1},\mu}(y) = \nu_\mu(y) = \mu_2$ by the

definition of $\nu^{(gh)^{-1},\mu}$. Since $B(\mathcal{O})$ is a subgroup of $\mathrm{GL}(2,\mathcal{O})$ we know that also $h^{-1} \in B(\mathcal{O})$ and thus $\nu((h^{-1})_c) \geq 1$ and $\nu((h^{-1})_a) = \nu((h^{-1})_d) = 0$. This implies that $\nu^{(gh)^{-1},\mu}((h^{-1})_a x) = \mu_1 < \mu_2 + 1 \leq \nu^{(gh)^{-1},\mu}((h^{-1})_c y)$ and thus $\nu^{gh,\mu}(h^{-1}x) = \nu_\mu(x)$. Similarly $\nu^{gh,\mu}(h^{-1}y) = \nu_\mu(y)$. As in the previous Lemma we can conclude that

$$\psi_{h^{-1}} : K_\mu\langle x, y, \rangle / J_{g^{-1}} \longrightarrow K_\mu\langle x, y\rangle / J_{(gh)^{-1}}$$

is an isometry and the Lemma follows as in the proof of Lemma 4.2.17. $\square$

Definition 4.2.20. Let V be a K-vector space and let ν be a semivaluation on V. Let $V_0 := \{v \in V : \nu(v) < \infty\}$. We define the map $\nu : V \times V_0 \longrightarrow \mathbb{R} \cup \{\infty\}$ by

$$\nu(f, g) := \nu(f) - \nu(g).$$

Moreover we define

$$H_1 := \mathcal{O}x + \mathcal{O}y - \pi(\mathcal{O}x + \mathcal{O}y) \subseteq K[x, y]_q.$$

Lemma 4.2.21. *Let $i, j \in \mathbb{R}$ and let $i < 0 < j$ and $g^{-1} \in G(q)$. Moreover let $\mu = (\mu_1, \mu_2)$ with $|\mu_1 - \mu_2| < 1$ and let $\mu^i = \mu + (i, i)$ and $\mu^j = \mu + (j, j)$. Then for every $s = s_1 \cdots s_n$ with $s_i \in H_1$ and every $f \in K[x, y]_q$ we have that*

$$\min\left\{\nu_{g,\mu^i}(f, s),\ \nu_{g,\mu^j}(f, s)\right\} \leq \nu_{g,\mu}(f, s).$$

Proof. We can assume that $s = s_1 \cdots s_n$ with $s_n \in gH_1$ since for $s' \in H_1$ there exists a $k \in \mathbb{Z}$ such that $\pi^k s' \in gH_1$. For $s_i \in gH_1$ we can write $s_i = s_i^1 gx + s_i^2 gy$ with $s_i^1, s_i^2 \in \mathcal{O}$ and $s_i^1 \in \mathcal{O}^\times$ or $s_i^2 \in \mathcal{O}^\times$. We have for $\eta \in \{\mu, \mu^i, \mu^j\}$ that

$$\nu_{g,\eta}(s_i) = \min\{\nu(s_i^1) + \eta_1, \nu(s_i^2) + \eta_2\}$$

Without loss of generality we will assume that $\mu_1 \leq \mu_2$. Moreover we define $k := \sharp\{i : \nu(s_i^1) \geq 1\}$. Then $|\mu_1 - \mu_2| < 1$ and Proposition 4.2.14 and the multiplicativity of $\nu_{g,\mu}$ imply

$$\nu_{g,\eta}(s) = (n - k)\eta_1 + k\eta_2$$

for $\eta \in \{\mu, \mu^i, \mu^j\}$. Let $f = \sum a_{c,d}(gx)^c(gy)^d$. Then there exist $r, t \in \mathbb{N}_0$ such that $\nu_{g,\mu}(f) = \nu(a_{r,t}) + \mu_1 r + \mu_2 t$. Then $\nu_{g,\mu^i}(f) \leq \nu(a_{r,t}) + (\mu_1 + i)r + (\mu_2 + i)t$.

If we assume that $r+t-n \geq 0$ i.e. $i(r+t-n) \leq 0$ then

$$\begin{aligned}\nu_{g,\mu}(f,s) &= \nu(a_{r,t}) + \mu_1 r + \mu_2 t - [(n-k)\mu_1 + k\mu_2] \\ &\geq \nu(a_{r,t}) + \mu_1 r + \mu_2 t - [(n-k)\mu_1 + k\mu_2] + i(r+t-n) \\ &= \nu(a_{r,t}) + (\mu_1+i)r + (\mu_2+i)t - [(n-k)(\mu_1+i) + k(\mu_2+i)] \\ &\geq \nu_{g,\mu^i}(f,s).\end{aligned}$$

If $r+t-n \leq 0$ one analogously shows $\nu_{g,\mu}(f,s) \geq \nu_{g,\mu^j}(f,s)$. □

Remark 4.2.22. Let $f \in K[x,y]_q$ and $s = s_1 \cdots s_n$ with $s_i \in H_1$. In order to construct the algebra attached to a finite combinatorial subtree we will later be interested in

$$\min\{\nu(f,s) : \nu \in M\}$$

for a certain set M of semivaluations on $K[x,y]_q$. The next lemma will ensure that out of the set $\left\{\nu_{g,\mu} : g^{-1} \in G(q), \epsilon_{r,\mu} > 0\right\}$ we only need to consider the valuations $\nu_{g,\mu}$ with $\mu_1 = \mu_2$ or $|\mu_1 - \mu_2| = 1$. These valuations will correspond to vertices of the tree.

Lemma 4.2.23. *Let $g^{-1} \in G(q)$ and let $|\mu_1 - \mu_2| < 1$.*

1. *Let $\mu_1 < \mu_2$. Let $\gamma > 0$ and $\delta > 0$ be such that*

$$\mu_1 \leq \mu_2 - \delta < \mu_2 < \mu_2 + \gamma \leq \mu_1 + 1$$

and $\epsilon_{g^{-1},(\mu_1,\mu_2+\gamma)} > 0$ and $\epsilon_{g^{-1},(\mu_1,\mu_2-\delta)} > 0$. Then for all $f \in K[x,y]_q$ and $s = s_1 \cdots s_n$ with $s_i \in H_1$ we have that

$$\min\{\nu_{g,(\mu_1,\mu_2+\gamma)}(f,s),\ \nu_{g,(\mu_1,\mu_2-\delta)}(f,s)\} \leq \nu_{g,\mu}(f,s).$$

2. *Let $\mu_1 < \mu_2$. Let $\gamma > 0$ and $\delta > 0$ be such that*

$$\mu_2 - 1 \leq \mu_1 - \delta < \mu_1 < \mu_1 + \gamma \leq \mu_2$$

and $\epsilon_{g^{-1},(\mu_1+\gamma,\mu_2)} > 0$ and $\epsilon_{g^{-1},(\mu_1-\delta,\mu_2)} > 0$. Then for all $f \in K[x,y]_q$ and s as above we have that

$$\min\{\nu_{g,(\mu_1+\gamma,\mu_2)}(f,s),\ \nu_{g,(\mu_1-\delta,\mu_2)}(f,s)\} \leq \nu_{g,\mu}(f,s).$$

Proof. We will only show the first statement as the second statement is proven similarly. As in the previous Lemma we can assume that $s_i \in gH_1$ and we can write $s_i = s_i^1 gx + s_i^2 gy$ with $s_i^1, s_i^2 \in \mathcal{O}$ and $s_i^1 \in \mathcal{O}^\times$ or $s_i^2 \in \mathcal{O}^\times$. Then

$$\nu_{g,\eta}(s_i) = \min\{\nu(s_i^1) + \eta_1, \nu(s_i^2) + \eta_2\}$$

for $\eta \in \{\mu, (\mu_1, \mu_2 + \gamma), (\mu_1, \mu_2 - \delta)\}$. Thus with $k := \sharp\{i : \nu(s_i^1) \geq 1\}$ we have that

$$\nu_{g,\eta}(s) = (n-k)\eta_1 + k\eta_2.$$

We have a presentation $f = \sum a_{c,d}(gx)^c(gy)^d$. Then there exist $r, t \in \mathbb{N}_0$ such that $\nu_{g,\mu}(f) = \nu(a_{r,t}) + r\mu_1 + t\mu_2$. We have that

$$\nu_{g,(\mu_1,\mu_2-\delta)}(f,s) \leq \nu(a_{r,t}) + r\mu_1 + t(\mu_2 - \delta) - [(n-k)\mu_1 + k(\mu_2 - \delta)].$$

Assume that $t - k \geq 0$. Then

$$\begin{aligned}
\nu_{g,\mu}(f,s) &= \nu(a_{r,t}) + r\mu_1 + t\mu_2 - [(n-k)\mu_1 + k\mu_2] \\
&= \nu(a_{r,t}) + r\mu_1 - (n-k)\mu_1 + (t-k)\mu_2 \\
&\geq \nu(a_{r,t}) + r\mu_1 - (n-k)\mu_1 + (t-k)(\mu_2 - \delta) \\
&= \nu(a_{r,t}) + r\mu_1 + t(\mu_2 - \delta) - [(n-k)\mu_1 + k(\mu_2 - \delta)] \\
&\geq \nu_{g,(\mu_1,\mu_2-\delta)}(f,s).
\end{aligned}$$

We also have that

$$\nu_{g,(\mu_1,\mu_1+\gamma)}(f,s) \leq \nu(a_{r,t}) + r\mu_1 + t(\mu_2 + \gamma) - [(n-k)\mu_1 + k(\mu_2 + \gamma)].$$

Assume that $t - k \leq 0$. Then

$$\begin{aligned}
\nu_{g,\mu}(f) - \nu_{g,\mu}(s) &= \nu(a_{r,t}) + r\mu_1 + t\mu_2 - [(n-k)\mu_1 + k\mu_2] \\
&= \nu(a_{r,t}) + r\mu_1 - (n-k)\mu_1 + (t-k)\mu_2 \\
&\geq \nu(a_{r,t}) + r\mu_1 - (n-k)\mu_1 + (t-k)(\mu_2 + \gamma) \\
&= \nu(a_{r,t}) + r\mu_1 + t(\mu_2 + \gamma) - [(n-k)\mu_1 + k(\mu_2 + \gamma)] \\
&\geq \nu_{g,(\mu_1,\mu_2+\gamma)}(f,s)
\end{aligned}$$

which concludes the proof. □

4.3 The Bruhat-Tits tree for $\mathrm{PGL}(2, K)$ and valuations on K^2

The Bruhat-Tits tree for $\mathrm{PGL}(2, K)$ has several descriptions and since two of them are useful for us we will shortly recall them. For a more detailed description see e.g. [DT08] Section 1.3.1, [Ser03] §1 or [RTW14] Chapter 1 and 2.
Moreover we will show that we can describe a subtree $\mathcal{S}$ of the Bruhat-Tits tree by a subset of $K^2 \times K^2 \times [0, 1)$. This will enable us in section 4.4 to attach a set of quasi abelian multiplicative semivaluations on $K[x, y]_q$ to $\mathcal{S}$.

4.3.1 (Bruhat-Tits tree)**.** An $\mathcal{O}$-lattice in K^2 is a finitely generated $\mathcal{O}$-submodule M of K^2 such that $M \otimes_{\mathcal{O}} K = K^2$. An $\mathcal{O}$-lattice in K^2 is always free of rank two. We denote the set of $\mathcal{O}$-lattices of K^2 by $\mathcal{L}_e$.
Two lattices M and M' are called homothetic if there exists an $a \in K^\times$ such that $M = aM'$. By $[M]$ we denote the homothety class of M. Let $\mathcal{L}$ be the set of homothety classes of $\mathcal{O}$-lattices in K^2.
Then $\mathcal{L}$ is the set of vertices $\mathcal{V}(\mathcal{T})$ of the Bruhat-Tits $\mathcal{T}$ tree associated to $\mathrm{PGL}(2, K)$. Two vertices $\mathfrak{m}$ and $\mathfrak{m}'$ are connected by an edge $e_{\mathfrak{m},\mathfrak{m}'}$ iff there exist $M \in \mathfrak{m}$ and $M' \in \mathfrak{m}'$ such that

$$\pi M \subsetneq M' \subsetneq M.$$

We will denote the set of edges by $\mathcal{E}(\mathcal{T})$. This construction yields a tree where every vertex is contained in exactly $\#\mathcal{O}/(\pi) + 1$ edges. In fact, for a vertex $[M]$ every one dimensional $\mathcal{O}/(\pi)$-vector space contained in $M/\pi M$ defines another vertex that is connected to $[M]$ by an edge.

4.3.2. On the vertices of the Bruhat-Tits tree $\mathcal{V}(\mathcal{T})$ one can define a distance d. For two vertices v_1, v_2 the distance $d(v_1, v_2)$ is equal to the length of the shortest path connecting v_1 and v_2. We then have that

$$d([M], [M']) := \min\{k \in \mathbb{N}_0 : \exists N \in [M'] \text{ with } M \supseteq N \supseteq \pi^k M\}.$$

For M, M' we can find a basis $\{b_1, b_2\}$ of K^2 such that $M = \mathcal{O}b_1 + \mathcal{O}b_2$ and $M' = \pi^l \mathcal{O}b_1 + \pi^m \mathcal{O}b_2$ for some $l, m \in \mathbb{Z}$. Then

$$d(M, M') = |m - l|,$$

see e.g. [Ser03] §1, 1.1. For $M, M' \in \mathcal{L}_e$ we define $d(M, M') := d([M], [M'])$.

4.3.3. There is also a description of the Bruhat-Tits tree by means of valuations on K^2. Let $\mathcal{N}_e$ denote the set of non-archimedean valuations on K^2. We call two valuations α, β equivalent if there exists a $c \in \mathbb{R}$ such that for all $v \in K^2$ we have that $\alpha(v) = \beta(v) + c$. Let $\mathcal{N}$ be the set of equivalence classes of valuations on K^2. For an $\alpha \in \mathcal{N}_e$ we denote its class by $[\alpha]$.

4.3.4. For $\alpha \in \mathcal{N}_e$ we can define a lattice $M_\alpha \in \mathcal{L}_e$ by

$$M_\alpha := \{v \in K^2 : \alpha(v) \geq 0\}.$$

Conversely a lattice $M \in \mathcal{L}_e$ defines a valuation $\alpha_M \in \mathcal{N}_e$ by

$$\alpha_M(v) := -\inf\{k \in \mathbb{Z} : \pi^k v \in M\}.$$

This induces bijections

$$\mathcal{L}_e \xleftrightarrow{1:1} \left\{\alpha \in \mathcal{N}_e : \alpha(K^2) \subseteq \mathbb{Z} \cup \{\infty\}\right\}$$

and

$$\mathcal{L} \xleftrightarrow{1:1} \left\{\mathfrak{n} \in \mathcal{N} : \exists \alpha \in \mathfrak{n} \text{ with } \alpha(K^2) \subseteq \mathbb{Z} \cup \{\infty\}\right\}.$$

4.3.5. By [RTW14] Proposition 1.20 for every $\alpha \in \mathcal{N}_e$ there exists a basis $b_1, b_2 \in K^2$ and $r_1, r_2 \in \mathbb{R}$ such that for all $a_1, a_2 \in K$ we have that

$$\alpha(a_1 b_1 + a_2 b_2) = \min\{\nu(a_1) + r_1, \nu(a_2) + r_2\}.$$

We define

$$\mathcal{N}_{\mathbb{Z}} := \left\{\alpha \in \mathcal{N}_e : \exists \beta \in [\alpha] \text{ with } \beta(K^2) \subseteq \mathbb{Z} \cup \{\infty\}\right\}.$$

For a valuation $\alpha \in \mathcal{N}_e$ with $\alpha \notin \mathcal{N}_{\mathbb{Z}}$ there exists a basis b_1, b_2 of K^2 and $\mu \in [0,1)^2$ with $\mu_1 \neq \mu_2$ such that $\alpha(a_1 b_1 + a_2 b_2) = \min\{\nu(a_1) + \mu_1, \nu(a_2) + \mu_2\}$ for all $a_1, a_2 \in K$.
Assume that $\mu_1 < \mu_2$ and that $\mu_1 = 0$. Then the lattices $[M] := [\mathcal{O}b_1 \oplus \mathcal{O}b_2]$ and $[M'] = [\mathcal{O}b_1 \oplus \mathcal{O}\pi^{-1}b_2]$ are connected by an edge and $\alpha_M \leq \alpha \leq \alpha_{M'}$. We say that $[\alpha]$ is contained in the edge between $[M]$ and $[M']$.
Conversely for every $\beta \in \mathcal{N}_e$ with $\alpha_M \leq \beta \leq \alpha_{M'}$ there exists $0 \leq r \leq 1$ such that $\beta(a_1 b_1 + a_2 b_2) = \min\{\nu(a_1), \nu(a_2) + r\}$ for all $a_1, a_2 \in K$. Thus the valuations lying between two adjacent vertices can be parametrized by $r \in [0,1]$ and we can view $\mathcal{N}$ as the geometric realization of the tree $\mathcal{T}$. This description together with the map d gives $\mathcal{N}$ the structure of a metric space. We will freely switch between the different descriptions of the Bruhat-Tits tree and it will be clear from the context if $\mathfrak{m} \in \mathcal{T}$ is an equivalence class of lattices or of valuations.

Definition 4.3.6. Let $M, M' \in \mathcal{L}_e$ be such that $M \subsetneq M' \subsetneq \pi^{-1}M$. Let $\{c_1, c_2\}$ be an $\mathcal{O}$ basis of M such that $\{c_1, \pi^{-1}c_2\}$ is an $\mathcal{O}$ basis of M' and let $0 \leq r < 1$. Then we can associate to M, M', r a valuation $\alpha_{M,M',r}$ on K^2 by setting

$$\alpha_{M,M',r}(a_1 c_1 + a_2 c_2) := \min\{\nu(a_1), \nu(a_2) + r\}.$$

This definition is independent of the choice of c_1 and c_2.

Definition 4.3.7. By a subtree of $\mathcal{T}$ we mean a closed connected subset of $\mathcal{N}$ which contains at least one vertex. A finite subtree is a subtree which contains only finitely many vertices. An open subtree of $\mathcal{T}$ is an open connected subset of $\mathcal{T}$. A combinatorial subtree of $\mathcal{T}$ is a closed connected subset $\mathcal{S}$ of $\mathcal{N}$ such that if for $e \in \mathcal{E}(\mathcal{T})$ there exists $a \in e - \mathcal{V}(\mathcal{T})$ with $a \in \mathcal{S}$, then we already have $e \subseteq \mathcal{S}$. Equivalently a combinatorial subtree $\mathcal{S} \subseteq \mathcal{T}$ is a subtree for which all

boundary points are vertices.
For a subtree $\mathcal{S}$ of $\mathcal{T}$ we denote by $\mathcal{S}^c$ the biggest combinatorial subtree contained in $\mathcal{S}$. Moreover we denote by $\mathcal{S}_m$ the smallest combinatorial subtree of $\mathcal{T}$ containing $\mathcal{S}$.
A subgraph of $\mathcal{T}$ is a closed subset of $\mathcal{T}$ such that every connected component is a subtree of $\mathcal{T}$.
For a set of vertices $\{v_1, ..., v_n\} \subseteq \mathcal{V}(\mathcal{T})$ we denote by $\langle v_1, ..., v_n\rangle$ the subgraph $\mathcal{S}$ of $\mathcal{T}$ with vertices $\mathcal{V}(\mathcal{S}) = \{v_1, ..., v_n\}$ and edges

$$\mathcal{E} = \{e \in \mathcal{T} : \exists v_i, v_j \in \{v_1, ..., v_n\} \text{ with } e = e_{v_i,v_j}\}.$$

4.3.8. We have a map

$$\begin{aligned} \mathrm{pr} : \mathcal{N}_e &\longrightarrow \mathcal{N} \\ \alpha &\longmapsto [\alpha]. \end{aligned}$$

For a vertex $v \in \mathcal{T}$ we will call $\mathrm{pr}^{-1}(v)$ an extended vertex and for an edge $e \in \mathcal{E}(\mathcal{T})$ we will call $\mathrm{pr}^{-1}(e)$ an extended edge. This gives $\mathcal{N}_e$ the structure of an extended Bruhat-Tits tree, see e.g. [Lan00] chapter 1.3. In [Lan00] chapter 1.3 an extended building is endowed with a metric. For the Bruhat-Tits tree we have isomorphisms of metric spaces $\mathrm{pr}^{-1}(v) \cong \mathbb{R}$ and $\mathrm{pr}^{-1}(e) \cong [0, 1] \times \mathbb{R}$ for $v \in \mathcal{V}(\mathcal{T})$ and $e \in \mathcal{E}(\mathcal{T})$. Thus the extended tree is a two dimensional metric space.
For a subtree $\mathcal{S} \subseteq \mathcal{T}$ we will denote the corresponding extended tree $\mathrm{pr}^{-1}(\mathcal{S})$ by $\mathcal{S}_e$ and the only extended trees that we will consider are of this form.

4.3.9. Let e_1, e_2 be the standard basis of K^2. For $\mu \in \mathbb{R}^2$ let $\alpha_\mu \in \mathcal{N}_e$ be defined by $\alpha_\mu(a_1e_1 + a_2e_2) := \{\nu(a_1) + \mu_1, \nu(a_2) + \mu_2\}$ for all $a_1, a_2 \in K$. For $g \in \mathrm{GL}(2, K)$ and $\mu \in \mathbb{R}^2$ we define the valuation $\alpha_{g,\mu} \in \mathcal{N}_e$ by

$$\alpha_{g,\mu}(m) := \alpha_\mu(g^{-1}m)$$

for all $m \in K^2$. We obtain a surjective map

$$\begin{aligned} \mathrm{GL}(2, K) \times [0, 1)^2 &\longrightarrow \mathcal{N}_e \\ (g, \mu) &\longmapsto \alpha_{g,\mu} \end{aligned}$$

which induces a bijection

$$\begin{aligned} \mathrm{GL}(2, K)/\,\mathrm{GL}(2, \mathcal{O}) \times [0, 1) &\longrightarrow \mathcal{N}_{\mathbb{Z}} \\ (\bar{g}, r) &\longmapsto \alpha_{g,(r,r)}. \end{aligned}$$

Recall that we denoted by $B(\mathcal{O})$ the Iwahori subgroup

$$B(\mathcal{O}) := \left\{ \begin{pmatrix} * & * \\ g_c & * \end{pmatrix} \in \mathrm{GL}(2, \mathcal{O}) : \nu(g_c) > 0 \right\}.$$

Let $A := \{(\mu_1, \mu_2) \in [0,1)^2 : \mu_1 > \mu_2\}$. Then we have a bijection

$$\begin{aligned} \mathrm{GL}(2, K)/B(\mathcal{O}) \times A &\longrightarrow \mathcal{N}_e - \mathcal{N}_{\mathbb{Z}} \\ (\bar{g}, \mu) &\longmapsto \alpha_{g,\mu}. \end{aligned}$$

Both bijections follow from [Par00] Corollary III.1.4.

Definition 4.3.10. Let $H_1 := \mathcal{O}^2 - \pi\mathcal{O}^2$. Then we define for $\alpha \in \mathcal{N}_e$ the tuple $\mathfrak{d}(\alpha) := (\mathfrak{d}_1(\alpha), \mathfrak{d}_2(\alpha)) \in \mathbb{R}^2$ by

$$\mathfrak{d}_1(\alpha) := \max\{\alpha(h) : h \in H_1\}$$
$$\mathfrak{d}_2(\alpha) := \min\{\alpha(h) : h \in H_1\}$$

This translates to $\mathcal{L}_e$ as follows. For $\mathfrak{m} \in \mathcal{L}$ we define $\mathfrak{m}_s \in \mathcal{L}_e$ to be the unique element in $\mathfrak{m}$ with $\mathfrak{m}_s \subseteq \mathcal{O}^2$ but $\mathfrak{m}_s \nsubseteq \pi\mathcal{O}^2$. Moreover we define $\mathfrak{m}_t \in \mathcal{L}_e$ to be the unique element in $\mathfrak{m}$ with $\mathfrak{m}_t \supseteq \mathcal{O}^2$ but $\pi\mathfrak{m}_t \nsupseteq \mathcal{O}^2$. For an element $N \in \mathfrak{m}$ the number $\mathfrak{d}_1(N) \in \mathbb{Z}$ is defined by the equation

$$N = \pi^{-\mathfrak{d}_1(N)}\mathfrak{m}_s$$

and $\mathfrak{d}_2(N) \in \mathbb{Z}$ is defined by the equation

$$N = \pi^{\mathfrak{d}_2(N)}\mathfrak{m}_t.$$

For $M \in \mathcal{L}_e$ we have that $\mathfrak{d}_i(M) = \mathfrak{d}_i(\alpha_M)$.

4.3.11. Let $\mathfrak{m} \in \mathcal{T}$ and let $f \in H_1 \cap \mathfrak{m}_s$. Then

$$\mathfrak{m}_s = \mathcal{O}f + \pi^{d([\mathcal{O}^2],\mathfrak{m})}\mathcal{O}^2.$$

Let moreover $g \in H_1$ be such that $\{f, g\}$ forms an $\mathcal{O}$-basis of $\mathcal{O}^2$. Then

$$\mathfrak{m}_s = \mathcal{O}f + \pi^{d([\mathcal{O}^2],\mathfrak{m})}\mathcal{O}g.$$

Both claims follow from the description of d in **4.3.1**.

Definition 4.3.12. For a subtree $\mathcal{S} \subseteq \mathcal{T}$ and $r_1, r_2 \in \mathbb{R}$ with $r_1 \geq r_2$ we define $\mathcal{S}_{r_1,r_2} \subseteq \mathcal{N}_e$ by

$$\mathcal{S}_{r_1,r_2} := \{\alpha \in \mathcal{L}_e : [\alpha] \in \mathcal{S} \text{ and } \mathfrak{d}_1(\alpha) \leq r_1 \text{ and } \mathfrak{d}_2(\alpha) \geq r_2\}.$$

Note that if $|r_1 - r_2|$ is small or if $\mathcal{S}$ is a infinite tree there might be edges or vertices in $\mathcal{S}$ such that there does not exists an $\alpha \in \mathcal{S}_{r_1,r_2}$ projecting on them. But for a sequence $\{(s_n, r_n)\}_{n\in\mathbb{N}}$ with $\lim\limits_{n\to\infty} s_n = \infty$ and $\lim\limits_{n\to\infty} r_n = -\infty$ we have

$$\mathcal{S}_e = \bigcup_{n\in\mathbb{N}} \mathcal{S}_{s_n,r_n}.$$

Lemma 4.3.13. *Let $M = \mathcal{O}^2$ and let $M_1 \subseteq M$ be such that $M_1 \not\subseteq \pi M$. Then for $f \in H_1$ we have that*

$$f \in M_1 + \pi^{\alpha_{M_1}(f)+d(M,M_1)} M.$$

Proof. Certainly $\alpha_{M_1}(f) \geq -d(M, M_1)$. If $\alpha_{M_1}(f) = -d(M, M_1)$ then the statement is obvious so we can assume that $\alpha_{M_1}(f) + d(M, M_1) > 0$. By **4.3.11** there exists a $b \in H_1$ such that $M_1 = \mathcal{O}b + \pi^{d(M,M_1)} M$. Then

$$\pi^{\alpha_{M_1}(f)} M_1 = \pi^{\alpha_{M_1}(f)} \mathcal{O}b + \pi^{\alpha_{M_1}(f)+d(M,M_1)} M.$$

Since $f \in \pi^{\alpha_{M_1}(f)} M_1$ we can write $f = a_f b + f_1$ with $a_f \in K$, $\nu(a_f) \geq \alpha_{M_1}(f)$ and $f_1 \in \pi^{\alpha_{M_1}(f)+d(M,M_1)} M$.
Assume that $\nu(a_f) < 0$. Then $\alpha_{M_1}(f) + d(M, M_1) > 0$ and $f = a_f b + f_1$ imply that $f \notin H_1$ which is a contradiction to our assumption. Hence $\nu(a_f) \geq 0$ and thus

$$f \in \mathcal{O}b + \pi^{\alpha_{M_1}(f)+d(M,M_1)} M \subseteq M_1 + \pi^{\alpha_{M_1}(f)+d(M,M_1)} M.$$

□

Definition 4.3.14. Recall that $H_1 := \mathcal{O}^2 - \pi\mathcal{O}^2$ and let $\mathcal{M} \subseteq \mathcal{N}_e$. We then define $A(\mathcal{M}) \subseteq K^2 \times H_1 \times [0, 1)$ by

$$A(\mathcal{M}) := \left\{(f, g, r) \in K^2 \times H_1 \times [0, 1) : \alpha(f) + r \geq \alpha(g) \text{ for all } \alpha \in \mathcal{M}\right\}.$$

For a subtree $\mathcal{S}$ of $\mathcal{T}$ we define

$$A(\mathcal{S}) := \left\{(f, g, r) \in K^2 \times H_1 \times [0, 1) : \alpha(f) + r \geq \alpha(g) \text{ for all } [\alpha] \in \mathcal{S}\right\}.$$

Lemma 4.3.15. *Recall from Definition 4.2.20 that for $\alpha \in \mathcal{N}_e$ we have that $\alpha(f, g) = \alpha(f) - \alpha(g)$ for $f, g \in K^2 - \{(0, 0)\}$. Let $\mathfrak{m}, \mathfrak{n} \in \mathcal{T}$ be two adjacent vertices and let $[\alpha] \in e_{\mathfrak{m},\mathfrak{n}}$. Then*

$$\alpha(f, g) \geq \min\{\alpha_{\mathfrak{m}_s}(f, g), \alpha_{\mathfrak{n}_s}(f, g)\}.$$

Proof. For $\beta \in [\alpha]$ we have that $\beta(f, g) = \alpha(f, g)$. Thus using description of $[\alpha] \in e_{\mathfrak{m},\mathfrak{n}}$ from **4.3.5** one can use the same strategy as in the proof of Lemma 4.2.23. □

Lemma 4.3.16. *Let $M = \mathcal{O}^2$ and let $\mathcal{S} = \langle \mathfrak{m}_1, ..., \mathfrak{m}_n \rangle$ be a finite subtree of $\mathcal{T}$ containing $[M]$. Then*

$$A(\mathcal{S}) \supseteq \{(a, b, 0) \in H_1^2 \times \{0\} : \forall i : (b \in (\mathfrak{m}_i)_s \Rightarrow a \in (\mathfrak{m}_i)_s)\}$$

Moreover if $[\alpha] \in \mathcal{T}$ and $\alpha(f) \geq \alpha(g)$ for all $(f, g, 0) \in A(\mathcal{S})$ then $[\alpha] \in \mathcal{S}$.

Proof. Since for $\alpha \in [\beta] \in \mathcal{N}$ and $(f, g) \in H_1^2$ we have that

$$\alpha(f) - \alpha(g) = \beta(f) - \beta(g)$$

we only have to verify the inequalities in the definition of $A(\mathcal{S})$ at a single member of each class.
Let $\mathcal{M} := \{(a, b, 0) \in H_1^2 \times \{0\} : \forall i : (b \in (\mathfrak{m}_i)_s \Rightarrow a \in (\mathfrak{m}_i)_s)\}$. We will show the inclusion $A(\mathcal{S}) \supseteq \mathcal{M}$ by using the following strategy. We will first show that for $[\alpha] \in \mathcal{V}(\mathcal{S})$ and $(f, g, 0) \in \mathcal{M}$ we have that $\alpha(f) \geq \alpha(g)$ by a case by case analysis. Therefore we can restrict ourselves to the cases of α_{M_i} for $M_i := (\mathfrak{m}_i)_s$. Then we will use Lemma 4.3.15 to conclude that $\alpha(f) \geq \alpha(g)$ for all $[\alpha] \in \mathcal{S}$.

Let $(f, g, 0) \in \mathcal{M}$. Because $[M] \in \mathcal{S}$ we know that there exists $l \in \{1, ..., n\}$ such that $M_l = M$ and thus $g \in M_l$. Let $[M_i] \in \mathcal{S}$ be a vertex with maximal distance to $[M]$ with the obstruction that $g \in M_i$. By the definition of $\mathcal{M}$ this means that also $f \in M_i$. Since $f, g \notin \pi M \supseteq \pi M_j$ for every j we know that $\alpha_{M_j}(f) \leq 0$ and $\alpha_{M_j}(g) \leq 0$.
Moreover we have that $g \in M_i = \mathcal{O}f + \pi^{d(M,M_i)}M$ and hence there exists an $a \in \mathcal{O}^\times$ such that $g - af \in \pi^{d(M,M_i)}M$. Since for $h \in M$ we have that $\alpha_{M_j}(h) \geq -d(M, M_j)$ we can conclude

$$\alpha_{M_j}(af - g) \geq d(M, M_i) - d(M, M_j) \tag{4.3.1}$$

With the help of this preparatory work we are now able to show that

$$\alpha_{M_j}(f) \geq \alpha_{M_j}(g)$$

for all $j \in \{1, ..., n\}$.
Case 1: Let $M_j \supseteq M_i$. We know $f, g \in M_j$ and thus $\alpha_{M_j}(f) \geq 0$ and $\alpha_{M_j}(g) \geq 0$. Since we already showed that $\alpha_{M_j}(f) \leq 0$ and $\alpha_{M_j}(g) \leq 0$ we can conclude that

$$\alpha_{M_j}(f) = \alpha_{M_j}(g) = 0.$$

Case 2: Let $M_j \not\supseteq M_i$ and let $d(M, M_j) \leq d(M, M_i)$. Then (4.3.1) implies that $\alpha_{M_j}(af - g) \geq 0$. Because $[M_i]$ is of maximal distance to $[M]$ with the condition that $g \in M_i$ and $M_j \not\supseteq M_i$ we can conclude that $g \notin M_j$ and hence $\alpha_{M_j}(g) < 0$. Using $\alpha_{M_j}(af - g) \geq 0$ we can conclude

$$\alpha_{M_j}(g) = \alpha_{M_j}(af) = \alpha_{M_j}(f).$$

Case 3: Let $M_j \not\supseteq M_i$ and $d(M, M_j) > d(M, M_i)$ and $M_i \not\supseteq M_j$. Then by Lemma 4.3.13 we know that $f \in M_j + \pi^{d(M,M_j)+\alpha_{M_j}(f)}M$. Thus we can find a $z \in M_j$ such that

$$z - f \in \pi^{d(M,M_j)+\alpha_{M_j}(f)}M. \tag{4.3.2}$$

If $d(M, M_j) + \alpha_{M_j}(f) > 0$ then $f \in H_1$ implies that z in H_1. Otherwise $d(M, M_j) + \alpha_{M_j}(f) = 0$ and we can choose $z \in M_j$ such that $z \in H_1$. Hence we can assume that $z \in H_1$ i.e. $M_j = \mathcal{O}z + \pi^{d(M,M_j)}M$. Assume that

$$d(M, M_j) + \alpha_{M_j}(f) \geq d(M, M_i). \tag{4.3.3}$$

Because of $M_i = \mathcal{O}f + \pi^{d(M,M_i)}M$ the equations (4.3.2) and (4.3.3) imply

$$M_i = \mathcal{O}z + \pi^{d(M,M_i)}M \supseteq \mathcal{O}z + \pi^{d(M,M_j)}M = M_j.$$

Hence by contraposition $M_i \not\supseteq M_j$ implies that $d(M, M_j)+\alpha_{M_j}(f) < d(M, M_i)$ i.e.

$$\alpha_{M_j}(f) < d(M, M_i) - d(M, M_j).$$

But by the estimate (4.3.1) we know that $\alpha_{M_j}(af - g) \geq d(M, M_i) - d(M, M_j)$ for some $a \in \mathcal{O}^\times$ and thus we can conclude that $\alpha_{M_j}(f) = \alpha_{M_j}(g)$.

Case 4: Let $M_j \not\supseteq M_i$ and let $d(M, M_j) > d(M, M_i)$ and $M_i \supseteq M_j$. Let $w \in H_1$ be such that $\mathcal{O}w + \pi^{d(M,M_j)}M = M_j$. Then $w - g \in \pi^{d(M,M_i)}M$ and thus $\alpha_{M_j}(g) \geq d(M, M_i) - d(M, M_j)$. Similarly we obtain the inequality $\alpha_{M_j}(f) \geq d(M, M_i) - d(M, M_j)$.

Since $\mathcal{S}$ is connected, $[M_i], [M_j] \in \mathcal{S}$ and $M_i \supseteq M_j$ we can conclude for $d(M, M_i) \leq l \leq d(M, M_j)$ that

$$M_j + \pi^l M \in \mathcal{S}. \tag{4.3.4}$$

By Lemma 4.3.13 we know that $g \in M_j + \pi^{\alpha_{M_j}(g)+d(M,M_j)}M$. Because of

$$d\left(M, M_j + \pi^{\alpha_{M_j}(g)+d(M,M_j)}M\right) = \alpha_{M_j}(g) + d(M, M_j)$$

the maximality of $d(M, M_i)$ and (4.3.4) imply that

$$d(M, M_i) \geq \alpha_{M_j}(g) + d(M, M_j).$$

Hence

$$\alpha_{M_j}(g) \leq d(M, M_i) - d(M, M_j) \leq \alpha_{M_j}(f).$$

Thus we showed that for $j \in \{1, ..., n\}$ we have that

$$\alpha_{M_j}(f) \geq \alpha_{M_j}(g)$$

for all $(f, g, 0) \in \mathcal{M}$.

Now let for $c, d \in \{1, ..., n\}$ be $[M_c], [M_d] \in \mathcal{S}$ two adjacent vertices and let $[\alpha] \in e_{[\alpha_{M_c}],[\alpha_{M_d}]}$.
Then we know by Lemma 4.3.15 that for all $f, g \in K^2 - \{(0,0)\}$

$$\alpha(f, g) \geq \min\{\alpha_{M_c}(f, g), \alpha_{M_d}(f, g)\}.$$

Since for $(f, g, 0) \in \mathcal{M}$ we showed that $\min\{\alpha_{M_c}(f, g), \alpha_{M_d}(f, g)\} \geq 0$ we can conclude that

$$\alpha(f) \geq \alpha(g)$$

for all $[\alpha] \in \mathcal{S}$ and thus $(f, g, 0) \in A(\mathcal{S})$.

We are left to show the last statement. We will show that for all $[\alpha] \notin \mathcal{S}$ there exists a $(f, g, 0) \in \mathcal{M} \subseteq A(\mathcal{S})$ such that $\alpha(f) < \alpha(g)$.
First assume that $[\alpha] = [M'] \notin \mathcal{S}$. We can assume that $M' \subseteq M$ but $M' \nsubseteq \pi M$. Let $g \in M' \cap H_1$. Since $[M] \in \mathcal{S}$ we know that for every $g \in M' \cap H_1$ there exists an $j \in \{1, ..., n\}$ such that $g \in M_j$. Let $[M_k] \in \mathcal{S}$ be of maximal distance to $[M]$ with $g \in M_k$. Since $\mathcal{S}$ is connected and contains $[M]$ there exists $i_k < i' \in \mathbb{N}_0$ and $f \in H_1$ such that $M_k = \mathcal{O}g + \pi^{i_k}\mathcal{O}f$ and $M' = \mathcal{O}g + \pi^{i'}\mathcal{O}f$. Because of the maximality of k we know by the first statement of the lemma that $(g + \pi^{i_k} f, g, 0) \in A(\mathcal{S})$. But since $g + \pi^{i_k} f \notin M'$ we know that

$$\alpha_{M'}(g + \pi^{i_k} f) < 0 = \alpha_{M'}(g).$$

Let $[\alpha] \notin \mathcal{S}$ and $\alpha \notin \mathcal{N}_{\mathbb{Z}}$. We can assume that $\mathfrak{d}_1(\alpha) = 0$. Then by **4.3.5** and Definition 4.3.6 there exists $\mathfrak{m}, \mathfrak{n} \in \mathcal{T}$ with $\mathfrak{m} \notin \mathcal{S}$ and such that $\alpha = \alpha_{\mathfrak{m}_s, \mathfrak{n}_s, r}$ for $r \in (0, 1)$. Here $\mathfrak{m}_s, \mathfrak{n}_s$ are as in Definition 4.3.10. There exist $f, g \in H_1$ and $l \geq 1$ such that $\mathfrak{m}_s = \mathcal{O}g + \pi^l\mathcal{O}f$ and $\mathfrak{n}_s = \mathcal{O}g + \pi^{l-1}\mathcal{O}f$. As above one can show that there exists $k \leq l - 1$ such that $(g + \pi^k f, g, 0) \in A(\mathcal{S})$. But

$$\alpha(g + \pi^k f) = \alpha_{\mathfrak{m}_s, \mathfrak{n}_s, r}(g + \pi^k f) = (k - l) + r < 0 = \alpha_{\mathfrak{m}_s, \mathfrak{n}_s, r}(g) = \alpha(g).$$

$\square$

Proposition 4.3.17. *Let $\mathcal{S}$ be a finite subtree of $\mathcal{T}$ containing $[M] = [\mathcal{O}^2]$ and let $[\alpha] \in \mathcal{T}$. Then $\alpha(f) + r \geq \alpha(g)$ for all $(f, g, r) \in A(\mathcal{S})$ implies that $[\alpha] \in \mathcal{S}$.*

Proof. Recall from Definition 4.3.7 that $\mathcal{S}_m$ is the smallest combinatorial subtree of $\mathcal{T}$ containing $\mathcal{S}$. Since $A(\mathcal{S}) \supseteq A(\mathcal{S}_m)$ we know by Lemma 4.3.16 that $\alpha(f)+r \geq \alpha(g)$ for all $(f,g,r) \in A(\mathcal{S})$ implies that $[\alpha] \in \mathcal{S}_m$. Let $[\alpha] \in \mathcal{S}_m - \mathcal{S}$. We can assume that $\mathfrak{d}_1(\alpha) = 0$. Then there exist $r \in (0,1)$ and $\mathfrak{m}, \mathfrak{n} \in \mathcal{T}$ such that the following holds

1. $d(\mathfrak{m}, \mathfrak{n}) = 1$ and $d([M], \mathfrak{n}) < d([M], \mathfrak{m})$
2. $\mathfrak{n} \in \mathcal{S}$ and $\mathfrak{m} \in \mathcal{S}_m - \mathcal{S}$
3. $\alpha = \alpha_{\mathfrak{m}_s, \mathfrak{n}_s, r}$.

Here $\mathfrak{m}_s$ and $\mathfrak{n}_s$ are as in Definition 4.3.10. Note that $\alpha_{\mathfrak{m}_s, \mathfrak{n}_s, 1} = \alpha_{\mathfrak{n}_s}$.
Let $z = \min\{a \in (0,1) : [\alpha_{\mathfrak{m}_s, \mathfrak{n}_s, a}] \in \mathcal{S}\} > r$. Note that $z \in \mathbb{R}$ exists since $\mathcal{S}$ is assumed to be closed. Let $d(\mathfrak{n}) := d([M], \mathfrak{n})$. Let $f, g \in H_1$ be such that $\mathfrak{n}_s = \mathcal{O}g + \pi^{d(\mathfrak{n})}\mathcal{O}f$ and $\mathfrak{m}_s = \mathcal{O}g + \pi^{d(\mathfrak{n})+1}\mathcal{O}f$.
Claim: $(g + \pi^{d(\mathfrak{n})}f, g, 1-z) \in A(\mathcal{S})$.
For $\mathfrak{l} \in \mathcal{T}$ with $g \in \mathfrak{l}_s$ we have that $\mathfrak{l}_s = \mathcal{O}g + \pi^{d(\mathfrak{l})}f\mathcal{O}$ by **4.3.11**. Thus since $\mathcal{S}$ is connected, $[M] \in \mathcal{S}$ and $\mathfrak{m} \notin \mathcal{S}$, we know that for $\mathfrak{l} \in \mathcal{S}_m - (e_{\mathfrak{m},\mathfrak{n}} - \{\mathfrak{n}\})$ with $g \in \mathfrak{l}_s$ we have that $d(\mathfrak{l}) \leq d(\mathfrak{n})$. Hence $\mathfrak{l} \in \mathcal{S}_m - (e_{\mathfrak{m},\mathfrak{n}} - \{\mathfrak{n}\})$ with $g \in \mathfrak{l}_s$ implies that $g + \pi^{d(\mathfrak{n})}f \in \mathfrak{l}_s$. Thus by Lemma 4.3.16 we know that

$$(g + \pi^{d(\mathfrak{n})}f, g, 0) \in A\left(\mathcal{S}_m - (e_{\mathfrak{m},\mathfrak{n}} - \{\mathfrak{n}\})\right)$$

and therefore also $(g+\pi^{d(\mathfrak{n})}f, g, 1-z) \in A\left(\mathcal{S}_m - (e_{\mathfrak{m},\mathfrak{n}} - \{\mathfrak{n}\})\right)$. Hence we only have show $\beta\left(g + \pi^{d(\mathfrak{n})}f\right) + 1 - z \geq \beta(g)$ for $\beta \in \mathcal{T}$ of the form $\beta = \alpha_{\mathfrak{m}_s,\mathfrak{n}_s,a}$ with $a \in [z, 1)$. But for such β we have that

$$\beta\left(g + \pi^{d(\mathfrak{n})}f\right) + 1 - z = -1 + a + 1 - z \geq 0 = \beta(g)$$

and thus $(g + \pi^{d(\mathfrak{n})}f, g, 1-z) \in A(\mathcal{S})$. This proves the claim.
But since

$$\alpha(g + \pi^{d(\mathfrak{n})}f) + 1 - z = \alpha_{\mathfrak{m}_s,\mathfrak{n}_s,r}(g + \pi^{d(\mathfrak{n})}f) + 1 - z = r - z < 0 = \alpha(g)$$

the Lemma is proven. □

Proposition 4.3.18. *Let $\mathcal{S}$ be a finite subtree of $\mathcal{T}$ and let $[\alpha] \in \mathcal{T}$. Then $\alpha(f) + r \geq \alpha(g)$ for all $(f,g,r) \in A(\mathcal{S})$ implies $[\alpha] \in \mathcal{S}$.*

Proof. Since for every finite subtree $\mathcal{S}$ there exists a $g \in \mathrm{GL}(2,K)$ such that $[\mathcal{O}^2] \in g\mathcal{S}$ the Proposition follows from Proposition 4.3.17. □

Proposition 4.3.19. *Let $\mathcal{S}$ be a finite subtree of $\mathcal{T}$. Let $r_1, r_2 \in \mathbb{R}$ with $r_1 \geq r_2$. If $\alpha \in \mathcal{N}_e$ and $\alpha(f) + r \geq \alpha(g)$ for all $(f,g,r) \in A(\mathcal{S})$ and $r_2 \leq \alpha(h) \leq r_1$ for all $h \in H_1$, then already $\alpha \in \mathcal{S}_{r_1,r_2}$.*

Proof. If for $\alpha \in \mathcal{N}_e$ we have $\alpha(f) + r \geq \alpha(g)$ for all $(f, g, r) \in A(\mathcal{S})$ then Lemma 4.3.16 implies $[\alpha] \in \mathcal{S}$. If in addition $r_2 \leq \alpha(h) \leq r_1$ for all $h \in H_1$ we can conclude that $\alpha \in \mathcal{S}_{r_1,r_2}$. □

4.4 The Bruhat-Tits tree and quasi abelian valuations on $K[x, y]_q$

Definition 4.4.1. Let

$$\mathcal{N}_{q,e} := \left\{\alpha \in \mathcal{N}_e : \alpha = \alpha_{g,\mu} \text{ for } g^{-1} \in G(q) \text{ and } \mu_1, \mu_2 \in \mathbb{R} \text{ with } |\mu_1 - \mu_2| < 1\right\}.$$

Here $\alpha_{g,\mu} \in \mathcal{N}_e$ is defined as in **4.3.9**. Let $\mathcal{N}_q$ be the image of $\mathcal{N}_{q,e}$ under the projection $\mathcal{N}_e \longrightarrow \mathcal{N}$.
Note that for $[\alpha] \in \mathcal{N}_q$ we have that $\{\beta \in \mathcal{N}_e : \beta \in [\alpha]\} \subseteq \mathcal{N}_{q,e}$ since $G(q)$ is stable under the action of

$$Z := \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in K^\times \right\} \subseteq \mathrm{GL}(2, K).$$

We define $\mathcal{T}_q$ to be the subgraph of $\mathcal{T}$ corresponding to $\mathcal{N}_q$.

4.4.2. There is an injective map

$$\mathcal{N}_{q,e} \longrightarrow \mathrm{Spmq}(K[x, y]_q)$$

and via this map we will view elements in $\mathcal{N}_{q,e}$ as elements in $\mathrm{Spmq}(K[x, y]_q)$. This map is defined as follows. Let $A := \{(\mu_1, \mu_2) \in [0, 1)^2 : \mu_1 > \mu_2\}$. By **4.3.9** there are bijections

$$\mathrm{GL}(2, K)/\mathrm{GL}(2, \mathcal{O}) \times [0, 1) \longrightarrow \mathcal{N}_{\mathbb{Z}}$$

and

$$\mathrm{GL}(2, K)/B(\mathcal{O}) \times A \longrightarrow \mathcal{N}_e \backslash \mathcal{N}_{\mathbb{Z}}.$$

The Lemmas 4.2.17 and 4.2.19 imply that we have a map

$$\begin{aligned} \mathcal{N}_{q,e} &\longrightarrow \mathrm{Spmq}(K[x, y]_q) \\ \alpha_{g,\mu} &\longmapsto \nu_{g,\mu}. \end{aligned}$$

This map is a section of the map $\mathrm{Spmq}(K[x, y]_q) \longrightarrow \mathcal{N}_{q,e}$ which sends an element $\mathfrak{a} \in \mathrm{Spmq}(K[x, y]_q)$ to $\mathfrak{a} \restriction_{K^2}$. Here we identified K^2 with the K-vector space $Kx \oplus Ky \subseteq K[x, y]_q$.

Lemma 4.4.3. *Recall that for $\mathfrak{m} \in \mathcal{V}(\mathcal{T})$ and $M = \mathcal{O}^2$ we defined the number $d(\mathfrak{m}) = d([M], \mathfrak{m})$.*

1. *Let $\mathfrak{m} \in \mathcal{V}(\mathcal{T})$ but $\mathfrak{m} \notin \mathcal{V}(\mathcal{T}_q)$. Then for every adjacent vertex $\mathfrak{m}'$ with $d(\mathfrak{m}') > d(\mathfrak{m})$ we have that $\mathfrak{m}' \notin \mathcal{V}(\mathcal{T}_q)$.*

2. *Let $\nu(1-q) \geq 2$ and let $\mathfrak{m}, \mathfrak{m}'$ be adjacent vertices with $d(\mathfrak{m}) < d(\mathfrak{m}')$ and $\mathfrak{m} \in \mathcal{V}(\mathcal{T}_q)$ but $\mathfrak{m}' \notin \mathcal{V}(\mathcal{T}_q)$. Then for every vertex $\mathfrak{m}'' \in \mathcal{V}(\mathcal{T})$ which is adjacent to $\mathfrak{m}$ and $d(\mathfrak{m}) < d(\mathfrak{m}'')$ we have that $\mathfrak{m}'' \notin \mathcal{V}(\mathcal{T}_q)$.*

Proof. **Statement 1:** We can choose an element $g \in \mathrm{GL}(2, \mathcal{O})$ such that $\mathfrak{m}_s = \mathcal{O}ge_1 + \pi^{d(\mathfrak{m})}ge_2$ and $\mathfrak{m}'_s = \mathcal{O}ge_1 + \pi^{d(\mathfrak{m})+1}ge_2$ where $\{e_1, e_2\}$ is the standard basis of $\mathcal{O}^2$. Let $I := \begin{pmatrix} 1 & 0 \\ 0 & \pi^{d(\mathfrak{m})} \end{pmatrix}$ and $I_1 := \begin{pmatrix} 1 & 0 \\ 0 & \pi^{d(\mathfrak{m}+1)} \end{pmatrix}$. Then $\mathfrak{m} = gI[M]$ and $\mathfrak{m}' = gI_1[M]$. We know that $g^{-1} \in G(q)$ since $\mathrm{GL}(2, \mathcal{O}) \subset G(q)$ and hence

$$\tau_{g^{-1}} = \nu(1-q) + \min\{\nu(g_a^{-1}), \nu(g_c^{-1})\} + \min\{\nu(g_b^{-1}), \nu(g_d^{-1})\} - \det g^{-1} > 0.$$

By slight abuse of notation we denote $(g^{-1})_x$ by g_x^{-1} for $x \in \{a, b, c, d\}$. Since

$$\begin{aligned} \tau_{I^{-1}g^{-1}} = \nu(1-q) &+ \min\{\nu(g_a^{-1}), \nu(g_c^{-1}) - d(\mathfrak{m}))\} \\ &+ \min\{\nu(g_b^{-1}), \nu(g_d^{-1}) - d(\mathfrak{m})\} - \det g^{-1} + d(\mathfrak{m}) \end{aligned}$$

the assumption $\mathfrak{m} = [\alpha_{gI}] \notin \mathcal{V}(\mathcal{T}_q)$ i.e. $\tau_{I^{-1}g^{-1}} \leq 0$ implies that

$$\min\{\nu(g_a^{-1}), \nu(g_c^{-1}) - d(\mathfrak{m})\} = \nu(g_c^{-1}) - d(\mathfrak{m})$$

and

$$\min\{\nu(g_b^{-1}), \nu(g_d^{-1}) - d(\mathfrak{m})\} = \nu(g_d^{-1}) - d(\mathfrak{m}).$$

This implies that also

$$\min\{\nu(g_a^{-1}), \nu(g_c^{-1}) - d(\mathfrak{m}) - 1\} = \nu(g_c^{-1}) - d(\mathfrak{m}) - 1$$

and

$$\min\{\nu(g_b^{-1}), \nu(g_d^{-1}) - d(\mathfrak{m}) - 1\} = \nu(g_d^{-1}) - d(\mathfrak{m}) - 1.$$

Hence

$$\begin{aligned} \tau_{I_1^{-1}g^{-1}} &= \nu(1-q) + \min\{\nu(g_a^{-1}), \nu(g_c^{-1}) - d(\mathfrak{m}) - 1\} \\ &\quad + \min\{\nu(g_b^{-1}), \nu(g_d^{-1}) - d(\mathfrak{m}) - 1\} - \det g^{-1} + d(\mathfrak{m}) + 1 \\ &= \tau_{I^{-1}g^{-1}} - 1 < 0 \end{aligned}$$

which proves the first statement.

Statement 2: Let g, I, I_1 be as in the first statement. Since $\mathfrak{m}' \notin \mathcal{V}(\mathcal{T}_q)$ i.e.

$\tau_{I_1^{-1}g^{-1}} \leq 0$ we can as in the proof of the first statement conclude that

$$\min\{\nu(g_a^{-1}), \nu(g_c^{-1}) - d(\mathfrak{m}')\} = \nu(g_c^{-1}) - d(\mathfrak{m}')$$
$$\min\{\nu(g_b^{-1}), \nu(g_d^{-1}) - d(\mathfrak{m}')\} = \nu(g_d^{-1}) - d(\mathfrak{m}')$$

and thus

$$\begin{aligned}0 &\geq \tau_{I_1^{-1}g^{-1}} \\ &= \nu(1-q) + \nu(g_c^{-1}) - d(\mathfrak{m}') + \nu(g_d^{-1}) - d(\mathfrak{m}') + d(\mathfrak{m}') \\ &\geq \nu(1-q) + \nu(g_c^{-1}) + \nu(g_d^{-1}) - d(\mathfrak{m}').\end{aligned}$$

Since we assumed that $\nu(1-q) \geq 2$ this means that $\nu(g_c^{-1}) \leq d(\mathfrak{m}') - 2$ and $\nu(g_d^{-1}) \leq d(\mathfrak{m}') - 2$. Since $\mathfrak{m}''_s \subset \mathfrak{m}_s$ and $\mathfrak{m}''$ and $\mathfrak{m}$ are adjacent we know that

$$\mathfrak{m}''_s = \mathcal{O}(gI_1e_1 + a\pi^{d(\mathfrak{m})}ge_2) + \mathcal{O}gI_1e_2$$

for some $a \in \mathcal{O}$. We have that $a\pi^{d(\mathfrak{m})}ge_2 = a\pi^{d(\mathfrak{m})}g\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}I_1e_1$ and

$$a\pi^{d(\mathfrak{m})}g\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}I_1e_2 = 0.$$

Hence for $\hat{g} = \left(g + a\pi^{d(\mathfrak{m})}g\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right)I_1$ we have that

$$\mathfrak{m}''_s = \mathcal{O}\hat{g}e_1 + \mathcal{O}\hat{g}e_2 = \hat{g}M.$$

Then for $\bar{g} := g + a\pi^{d(\mathfrak{m})}g\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$ we know that $\bar{g} \in \mathrm{GL}(2, \mathcal{O})$. Because of $\nu(g_c^{-1}) \leq d(\mathfrak{m}') - 2 = d(\mathfrak{m}) - 1$ we can conclude that

$$\nu(\bar{g}_c^{-1}) = \nu\left(\frac{-\bar{g}_c}{\det(\bar{g})}\right) = \nu\left(\bar{g}_c\right) = \nu\left(g_c + a\pi^{d(\mathfrak{m})}g_d\right) = \nu(g_c) = \nu(g_c^{-1}).$$

Similarly from $\nu(g_d^{-1}) \leq d(\mathfrak{m}') - 2 = d(\mathfrak{m}) - 1$ we obtain $\nu(\bar{g}_d^{-1}) = \nu(g_d^{-1})$. This implies that $\tau_{\hat{g}^{-1}} = \tau_{I_1^{-1}g^{-1}} \leq 0$ and hence $\mathfrak{m}'' \notin \mathcal{V}(\mathcal{T}_q)$. □

Corollary 4.4.4. *The subtree $\mathcal{T}_q$ is an infinite open subtree of $\mathcal{T}$ containing $[\mathcal{O}^2]$.*

Proof. That $\mathcal{T}_q$ is open and connected follows from Lemma 4.4.3 and the definition of $\mathcal{T}_q$. Since by Lemma 4.2.5 the set $G(q)$ is stable under all $\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$ with $a, b \in K^\times$ it is an infinite tree. □

Definition 4.4.5. Let $R := K[x,y]_q$ and let $r_1, r_2 \in \mathbb{R}$ with $r_1 \geq r_2$. For a subset $T \subset R \times (R - \{0\}) \times [0,1)$ we define

$$U(T) := \{\gamma \in \mathrm{Spmq}(R) : \gamma(a_1) + r \geq \gamma(a_2) \neq \infty \text{ for all } (a_1, a_2, r) \in T\}$$
$$U_{r_1,r_2}(T) := \{\gamma \in U(T) : r_2 \leq \gamma(a_2) \leq r_1 \text{ for all } (a_1, a_2, r) \in T\}.$$

4.4.6. Recall that we defined $H_1 = \mathcal{O}^2 - \pi\mathcal{O}^2$. By abuse of notation we also defined $H_1 \subseteq K[x,y]_q$ by

$$H_1 := (\mathcal{O}x + \mathcal{O}y) - \pi\,(\mathcal{O}x + \mathcal{O}y).$$

Note that there are a canonical isomorphism

$$\begin{array}{ccc} \mathcal{O}^2 - \pi\mathcal{O}^2 & \xrightarrow{\sim} & (\mathcal{O}x + \mathcal{O}y) - \pi\,(\mathcal{O}x + \mathcal{O}y) \\ (a,b) & \longmapsto & ax + by \end{array}$$

and a canonical embedding

$$\begin{array}{ccc} K^2 \times H_1 \times [0,1) & \hookrightarrow & K[x,y]_q \times K[x,y]_q - \{0\} \times [0,1)\,. \\ (a,b) \times (c,d) \times r & \longmapsto & (ax+by) \times (cx+dy) \times r \end{array}$$

Thus for $\mathcal{S} \subseteq \mathcal{T}$ we can view $A(\mathcal{S})$ as subset of $K[x,y]_q \times K[x,y]_q - \{0\} \times [0,1)$ and hence the expressions $U(A(\mathcal{S}))$ and $U_{r_1,r_2}(A(\mathcal{S}))$ make sense.

Lemma 4.4.7. *Let $\mathcal{S}$ be a finite subtree of $\mathcal{T}_q$. Then for every $\mathfrak{a} \in U(A(\mathcal{S}))$ there exists an $[\alpha] \in \mathcal{S}$ such that*

$$\mathfrak{a}(f) = \alpha(f)$$

for all $f \in H := Kx \oplus Ky \subseteq K[x,y]_q$.

Proof. The statement of the Lemma does only involve the restriction of $\mathfrak{a}$ to the two dimensional K-vector space $H = Kx \oplus Ky$. Thus the statement is a statement about semivaluations on a two dimensional K-vector space. Because $\mathrm{GL}(2,K)$ acts on such a vector space as automorphism, we may and will assume that $[\mathcal{O}^2] \in \mathcal{S}$.

Claim: Every $\mathfrak{a} \in U(A(\mathcal{S}))$ restricts to a valuation on H.

Let $f \in H_1 = \mathcal{O}x \oplus \mathcal{O}y - \pi(\mathcal{O}x \oplus \mathcal{O}y)$. Certainly $(f,f,0) \in A(\mathcal{S})$ and thus $\mathfrak{a}(f) \neq \infty$. This implies the claim.

Then we can find an $\alpha \in \mathcal{N}_e$ such that $\alpha = \mathfrak{a}\restriction_H$. But this means that $\alpha(f) + r \geq \alpha(g)$ for all $(f,g,r) \in A(\mathcal{S})$. By Proposition 4.3.17 this implies that $[\alpha] \in \mathcal{S}$. ☐

Definition 4.4.8. Similarly as in the case of valuations, cf. Definition 4.2.20, we will define for an element $\mathfrak{a} \in \mathrm{Spmq}(K[x,y]_q)$ and $f,g \in K[x,y]_q$ with

$\mathfrak{a}(g) \neq \infty$

$$\mathfrak{a}(f,g) := \mathfrak{a}(f) - \mathfrak{a}(g).$$

Lemma 4.4.9. *Assume that* $\mathfrak{a} \in \mathrm{Spmq}(K[x,y]_q)$ *coincides with* $\alpha \in \mathcal{N}_{q,e}$ *on* H*. Let* $s = s_1 \cdots s_n$ *with* $s_i \in H_1$ *and* $f \in K[x,y]_q$*. Then we have that*

$$\mathfrak{a}(f,s) \geq \alpha(f,s).$$

Proof. We know that there exists a $g^{-1} \in G(q)$ and $\mu \in \mathbb{R}^2$ such that $\alpha = \alpha_{g,\mu}$ i.e.

$$\alpha\left(\sum a_{i,j}(gx)^i(gy)^j\right) = \min\{\nu(a_{i,j}) + \mu_1 i + \mu_2 j\}$$

by Lemma 4.2.14. Because $\mathfrak{a}$ coincides with α on H we have that

$$\mu_1 = \alpha(gx) = \mathfrak{a}(gx)$$

and also $\mu_2 = \mathfrak{a}(gy)$. By the multiplicativity we thus obtain

$$\mathfrak{a}\left(\sum a_{i,j}(gx)^i(gy)^j\right) \geq \min\{\nu(a_{i,j}) + \mu_1 i + \mu_2 j\}.$$

Since we know by Lemma 4.2.10 that $\{(gx)^i(gy)^j : i,j \in \mathbb{N}_0\}$ is a basis of $K[x,y]_q$ and since $\mathfrak{a}(s) = \alpha(s)$ the Lemma is proven. $\square$

Definition 4.4.10. Let $W \subseteq \mathrm{Spmq}(K[x,y]_q)$ and let $\langle H_1 \rangle \subseteq K[x,y]_q$ be the multiplicative subset generated by H_1 . Assume that $\mathfrak{a}(s) \neq \infty$ for all $\mathfrak{a} \in W$ and all $s \in \langle H_1 \rangle$. We say that a set $\Lambda \subseteq \mathrm{Spmq}(K[x,y]_q)$ dominates W if for every $\mathfrak{a} \in W$ and every $f \in K[x,y]_q$ and every finite product $s \in \langle H_1 \rangle$ there exists a $\mathfrak{b} \in \Lambda$ such that

$$\mathfrak{a}(f,s) \geq \mathfrak{b}(f,s).$$

If Λ is a finite set we have that

$$l_\Lambda := \max\{r \in \mathbb{R} : \forall \mathfrak{b} \in \Lambda \text{ it is true that } \mathfrak{b} \text{ is quasi abelian of level } r\}$$

exists and $l_\Lambda > 0$. In this case we say that W is finitely dominated of level l_Λ by Λ.

4.4.11. Let $\mathfrak{m} \in \mathcal{V}(\mathcal{T}_q)$. Then there exists an $\mathcal{O}$-basis

$$\{f = (f_1, f_2), h = (h_1, h_2)\}$$

of $\mathcal{O}^2$ such that $\mathfrak{m}_s = \left[\mathcal{O}f + \pi^{d([\mathcal{O}^2],\mathfrak{m})}\mathcal{O}h\right]$. Let $g = \begin{pmatrix} f_1 & \pi^{d([\mathcal{O}^2],\mathfrak{m})}h_1 \\ f_2 & \pi^{d([\mathcal{O}^2],\mathfrak{m})}h_2 \end{pmatrix}$. Since $\mathfrak{m} \in \mathcal{T}_q$ we can conclude that $g^{-1} \in G(q)$ and every $\mathfrak{a} \in \mathfrak{m}$ is of the form

$$\mathfrak{a} = \nu_{g,(r,r)}$$

for some $r \in \mathbb{R}$. Since $\left\{gx, \pi^{-d([\mathcal{O}^2],\mathfrak{m})}gy\right\}$ is an $\mathcal{O}$-basis of $H = \mathcal{O}x + \mathcal{O}y$ we have that

$$\max\{\nu_{g,(r,r)}(z) : z \in H_1\} = r$$
$$\min\{\nu_{g,(r,r)}(z) : z \in H_1\} = r - d([\mathcal{O}^2], \mathfrak{m})$$

Let now $v \in \mathcal{T}_q - \mathcal{V}(\mathcal{T}_q)$. Similarly as above there exist a $g^{-1} \in G(q)$ and an $s \in (0,1)$ such that $\left\{gx, \pi^{-d([\mathcal{O}^2],[g\mathcal{O}^2])}gy\right\}$ is an $\mathcal{O}$-basis of H and every $\mathfrak{a} \in v$ is of the form $\mathfrak{a} = \nu_{g,(r,r+s)}$ for some $r \in \mathbb{R}$. In this case we have that $d([\mathcal{O}^2], [g\mathcal{O}^2]) \geq 1$. Then

$$\max\{\nu_{g,(r,r+s)}(z) : z \in H_1\} = r$$
$$\min\{\nu_{g,(r,r+s)}(z) : z \in H_1\} = r + s - d([\mathcal{O}^2], [g\mathcal{O}^2]).$$

Definition 4.4.12. Let $v \in \mathcal{V}(\mathcal{T}_q)$ and let $g^{-1} \in G(q)$ be as in **4.4.11**. We define for $r_1 \geq r_2$ such that $r_1 - r_2 \geq d([\mathcal{O}^2], v)$ the set

$$\mathfrak{B}_{r_1,r_2}(v) := \left\{\nu_{g,(r_1,r_1)}, \nu_{g,\left(r_2+d([\mathcal{O}^2],v), r_2+d([\mathcal{O}^2],v)\right)}\right\}.$$

For $v \in \mathcal{T}_q - \mathcal{V}(\mathcal{T}_q)$ and $g^{-1} \in G(q)$ and $s \in (0,1)$ as in **4.4.11** we define for $r_1 - r_2 \geq d([\mathcal{O}^2], v) - s$ the set

$$\mathfrak{B}_{r_1,r_2}(v) := \left\{\nu_{g,(r_1,r_1+s)}, \nu_{g,\left(r_2+d([\mathcal{O}^2],[g\mathcal{O}^2])-s, r_2+d([\mathcal{O}^2],[g\mathcal{O}^2])\right)}\right\}.$$

Definition 4.4.13. By **4.3.5** the set $\mathcal{N}$ has the structure of a metric space. For a subtree $\mathcal{S}$ we denote by $\mathcal{B}(\mathcal{S})$ the set of boundary points.

Definition 4.4.14. For a finite subtree $\mathcal{S} \subseteq \mathcal{T}_q$ and $r_1 \geq r_2$ such that the canonical projection $\mathcal{S}_{r_1,r_2} \to \mathcal{S}$ is surjective we define

$$\Lambda_{\mathcal{S}_{r_1,r_2}} := \bigcup_{v \in \mathcal{V}(S)} \mathfrak{B}_{r_1,r_2}(v) \cup \bigcup_{v \in \mathcal{B}(\mathcal{S})} \mathfrak{B}_{r_1,r_2}(v).$$

Then

$$l_{\Lambda_{\mathcal{S}_{r_1,r_2}}} = \max\left\{r \in \mathbb{R} : \ \mathfrak{b} \text{ is quasi abelian of level } r \ \forall \mathfrak{b} \in \Lambda_{\mathcal{S}_{r_1,r_2}}\right\}.$$

Proposition 4.4.15. *Let $\mathcal{S} \subset \mathcal{T}_q$ be a finite subtree and let $r_1 \geq r_2$ be such that the canonical projection $\mathcal{S}_{r_1,r_2} \to \mathcal{S}$ is surjective. Then the set $U_{r_1,r_2}(A(\mathcal{S}))$ is finitely dominated of level $l_{\Lambda_{\mathcal{S}_{r_1,r_2}}}$ by the finite set $\Lambda_{\mathcal{S}_{r_1,r_2}}$.*

Proof. Let $\mathfrak{a} \in U_{r_1,r_2}(A(\mathcal{S}))$. By Lemma 4.4.7 there exists an $[\alpha] \in \mathcal{S}$ such that $\mathfrak{a} \restriction_H = \alpha \restriction_H$. Since $r_1 \geq \mathfrak{a}(h) \geq r_2$ for all $h \in H$ we can conclude that $\alpha \in \mathcal{S}_{r_1,r_2}$. By Lemma 4.4.9 the quasi abelian semivaluation $\mathfrak{a}$ is dominated

by α. Moreover we surely have $\mathcal{S}_{r_1,r_2} \subseteq U_{r_1,r_2}(A(\mathcal{S}))$.
Thus it is enough to show that $\mathcal{S}_{r_1,r_2}$ is dominated by $\Lambda_{\mathcal{S}_{r_1,r_2}}$.
Case 1: Let $\alpha \in \mathcal{S}_{r_1,r_2}$ with $[\alpha] \in \mathcal{V}(\mathcal{S})$. Then there exist $g^{-1} \in G(q)$ and $r \in \mathbb{R}$ as in **4.4.11** such that $\alpha = \nu_{g,(r,r)}$ and

$$\max\{\nu_{g,(r,r)}(z) : z \in H_1\} = r$$
$$\min\{\nu_{g,(r,r)}(z) : z \in H_1\} = r - d([\mathcal{O}^2], [\alpha]).$$

Thus $r_1 \geq r \geq r - d([\mathcal{O}^2], [\alpha]) \geq r_2$. By Lemma 4.2.21 $\nu_{g,(r,r)}$ is dominated by the set

$$\left\{\nu_{g,(r_1,r_1)}, \nu_{g,\left(r_2+d([\mathcal{O}^2],[\alpha]),r_2+d([\mathcal{O}^2],[\alpha])\right)}\right\} \subseteq \Lambda_{\mathcal{S}_{r_1,r_2}}.$$

Case 2: Let $\alpha \in \mathcal{S}_{r_1,r_2}$ be such that $[\alpha] \in \mathcal{B}(\mathcal{S}) - \mathcal{V}(\mathcal{S})$. Then there exist $g^{-1} \in G(q)$ and $r \in \mathbb{R}$ and $s \in (0,1)$ as in **4.4.11** such that $\alpha = \nu_{g,r,r+s}$ and

$$\max\{\nu_{g,(r,r+s)}(z) : z \in H_1\} = r$$
$$\min\{\nu_{g,(r,r+s)}(z) : z \in H_1\} = r + s - d([\mathcal{O}^2], [g\mathcal{O}^2])$$

As in Case 1we can use Lemma 4.2.21 to conclude that $\nu_{g,(r,r+s)}$ is dominated by

$$\left\{\nu_{g,(r_1,r_1+s)}, \nu_{g,\left(r_2+d([\mathcal{O}^2],[g\mathcal{O}^2])-s,r_2+d([\mathcal{O}^2],[g\mathcal{O}^2])\right)}\right\} \subseteq \Lambda_{\mathcal{S}_{r_1,r_2}}.$$

Case 3: Let $\alpha \in \mathcal{S}_{r_1,r_2}$ be such that $[\alpha] \in \mathcal{S} - (\mathcal{B}(\mathcal{S}) \cup \mathcal{V}(\mathcal{S}))$. As in **Case 2** there exist $g^{-1} \in G(q)$ and $r \in \mathbb{R}$ and $s \in (0,1)$ as in **4.4.11** such that $\alpha = \nu_{g,r,r+s}$. Moreover there exist $0 \leq s_1 \leq s \leq s_2 \leq 1$ such that we have $[\nu_{g,(r,r+s_1)}] \in (\mathcal{B}(\mathcal{S}) \cup \mathcal{V}(\mathcal{S}))$ and $[\nu_{g,(r,r+s_2)}] \in (\mathcal{B}(\mathcal{S}) \cup \mathcal{V}(\mathcal{S}))$.
Let $v = [g\mathcal{O}^2]$. As in **Case 2** we can conclude that $\nu_{g,(r,r+s)}$ is dominated by

$$\left\{\nu_{g,(r_1,r_1+s)}, \nu_{g,\left(r_2+d([\mathcal{O}^2],v)-s,r_2+d([\mathcal{O}^2],v)\right)}\right\} \subseteq \mathcal{S}_{r_1,r_2}$$

By Lemma 4.2.23 we have that $\nu_{g,(r_1,r_1+s)}$ is dominated by

$$\left\{\nu_{g,(r_1,r_1+s_1)}, \nu_{g,(r_1,r_1+s_2)}\right\}.$$

Using Definition 4.4.12 we can conclude that

$$\left\{\nu_{g,(r_1,r_1+s_1)}, \nu_{g,(r_1,r_1+s_2)}\right\} \subseteq \Lambda_{\mathcal{S}_{r_1,r_2}}.$$

Lemma 4.2.23 also implies that $\nu_{g,\left(r_2+d([\mathcal{O}^2],v)-s,r_2+d([\mathcal{O}^2],v)\right)}$ is dominated by

$$\left\{\nu_{g,\left(r_2+d([\mathcal{O}^2],v)-s_1,r_2+d([\mathcal{O}^2],v)\right)}, \nu_{g,\left(r_2+d([\mathcal{O}^2],v)-s_2,r_2+d([\mathcal{O}^2],v)\right)}\right\} \subseteq \Lambda_{\mathcal{S}_{r_1,r_2}}$$

where the inclusion again follows from Definition 4.4.12. $\square$

4.5 Algebras attached to extended subtrees

In order to attach topological algebras to subsets of $\mathcal{N}_{q,e}$ we will use the theory of algebraic microlocalization as developed by P. Schneider in [Záb12].

Proposition 4.5.1 ([Záb12] Proposition A.18)**.** *Let A be a K-algebra and let M be a finite set of multiplicative quasi abelian valuations on A. Let S be a multiplicative subset of A. Then there exists a K-Banach algebra*

$$A\langle M, S\rangle$$

with submultiplicative valuation ν_M and a map $\eta : A \to A\langle M, S\rangle$ satisfying the following properties:

1. $\eta(s) \in A\langle M, S\rangle^\times$ *for all* $s \in S$.
2. $\nu_M\left(\eta(s)^{-1}\eta(a)\right) = \min\{\mathfrak{a}(a) - \mathfrak{a}(s) : \mathfrak{a} \in M\}$ *for all* $s \in S$, $a \in A$.
3. $\eta : A \to A\langle M, S\rangle$ *fulfills the following universal property:*

   Let (D, ν_D) be a K-Banach algebra and let $\phi : A \longrightarrow D$ be a morphism of K-algebras such that

   (a) $\phi(s) \in D^\times$ *for all* $s \in S$

   (b) *there exists a* $\gamma \in \mathbb{R}$ *such that*

   $$\nu_D\left(\phi(s)^{-1}\phi(a)\right) \geq \min\{\mathfrak{a}(a) - \mathfrak{a}(s) : \mathfrak{a} \in M\} + \gamma$$

   for all $a \in A$, $s \in S$.

   Then there exists a unique morphism of K-Banach algebras

   $$\phi_S : A\langle M, S\rangle \longrightarrow D$$

   such that $\phi_S \circ \eta = \phi$. If γ can be chosen as $\gamma = 0$, then ϕ_S is valuation increasing.

The K-Banach algebra $A\langle M, S\rangle$ is called the algebraic microlocalization of A at S with respect to M.

4.5.2. We will often omit the map η and for an element $f \in A$ we will write $f \in A\langle M, S\rangle$ instead of $\eta(f) \in A\langle M, S\rangle$.

Proposition 4.5.3 ([Záb12] Lemma A.16)**.** *Let A, M, S be as in Proposition 4.5.1. Then $\{s^{-1}f : s \in S, f \in A\} \subseteq A\langle M, S\rangle$ is a dense subset.*

Proposition 4.5.4 ([Záb12] Proposition A.21). *Let A, M, S be as in Proposition 4.5.1 and let $\gamma > 0$ be such that $\mathfrak{b} \in M$ is quasi abelian of level γ for all $\mathfrak{b} \in M$. For $e_1, ..., e_n \in A\langle M, S\rangle$ and $\sigma \in S_n$ we then have that*

$$\nu_M(e_1 \cdots e_n - e_{\sigma(1)} \cdots e_{\sigma(n)}) \geq \nu_M(e_1 \cdots e_n) + \gamma.$$

Definition 4.5.5. Let $\langle H_1\rangle = \{s_1 \cdots s_n : s_i \in H_1 \text{ and } n \in \mathbb{N}_0\}$. Let $\mathcal{S}$ be a finite subtree of $\mathcal{T}_q$ and let $r_1 \geq r_2$ be such that the canonical map $\mathcal{S}_{r_1,r_2} \to \mathcal{S}$ is surjective. Let $\Lambda_{\mathcal{S}_{r_1,r_2}}$ be as in Definition 4.4.14. Note that by definition $\Lambda_{\mathcal{S}_{r_1,r_2}}$ is a finite set of quasi abelian multiplicative valuations. We then define the K-Banach algebra $\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_{r_1,r_2})$ by

$$\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_{r_1,r_2}) := K[x,y]_q\langle\Lambda_{\mathcal{S}_{r_1,r_2}}, \langle H_1\rangle\rangle.$$

We will denote by $\nu_{\mathcal{S}_{r_1,r_2}}$ canonical valuation on $\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_{r_1,r_2})$.

Definition 4.5.6. For a K-Banach algebra A with valuation ν_A we define

$$\mathrm{Spmqb}(A) := \{\mathfrak{a} \in \mathrm{Spmq}(A) : \mathfrak{a}(a) \geq \nu_A(a) \text{ for all } a \in A\}.$$

Proposition 4.5.7. *Let $\mathcal{S} \subseteq \mathcal{T}_q$ be a finite subtree and let $r \in \mathbb{R}^2$ be such that $r_1 \geq r_2$ and that $\mathcal{S}_r \to \mathcal{S}$ is surjective. Then every $\mathfrak{a} \in U_r(\mathcal{S})$ extends to a multiplicative quasi abelian semivaluation on $\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_r)$ of the same level. Hence we obtain a map*

$$\varphi : U_r(\mathcal{S}) \longrightarrow \mathrm{Spmqb}\left(\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_r)\right)$$

which preserves the level.

Proof. Let $\mathfrak{a} \in U_r(\mathcal{S})$. Denote by $\widehat{K[x,y]}_q$ be the completion of $K[x,y]_q$ with respect to $\mathfrak{a}$. Hence $\mathfrak{a}$ defines a quasi abelian multiplicative valuation on $\widehat{K[x,y]}_q$. Since for $s \in \langle H_1\rangle$ we know that $\mathfrak{a}(s) \neq \infty$, we can conclude that $\langle H_1\rangle$ is a multiplicative subset in $\widehat{K[x,y]}_q$. Then we have the canonical map

$$\eta : K[x,y]_q \longrightarrow \widehat{K[x,y]}_q \longrightarrow \widehat{K[x,y]}_q\langle\{\mathfrak{a}\}, \langle H_1\rangle\rangle.$$

By Proposition 4.4.15 we have for $f \in K[x,y]$, $s \in \langle H_1\rangle$ that

$$\nu_{\{\mathfrak{a}\}}\left(\eta\left(s^{-1}\right)\eta(f)\right)) = \mathfrak{a}(f) - \mathfrak{a}(s) \geq \min\{\mathfrak{b}(f) - \mathfrak{b}(s) : \mathfrak{b} \in \Lambda_{\mathcal{S}_r}\} = \nu_{\mathcal{S}_r}\left(s^{-1}f\right) \tag{4.5.1}$$

Thus by the universal property of $K[x,y]_q\langle\Lambda_{\mathcal{S}_r}, \langle H_1\rangle\rangle$ we can conclude that there is a K-Banach algebra morphism

$$\phi : \mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_r) \longrightarrow \widehat{K[x,y]}_q\langle\{\mathfrak{a}\}, \langle H_1\rangle\rangle.$$

Thus $\nu_{\{\mathfrak{a}\}}$ induces a multiplicative quasi abelian semivaluation $\varphi(\mathfrak{a})$ on $\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_r)$. For $a \in K[x,y]_q$ we have that $\varphi(\mathfrak{a})(a) = \mathfrak{a}(a)$. We know that $\nu_{\{\mathfrak{a}\}}$ is quasi abelian of the same level as $\mathfrak{a}$ by [Záb12] Proposition A.21.
Let $\mathfrak{a}$ be of level γ. Then for $s, t \in \langle H_1 \rangle$ and $a, b \in K[x,y]_q$ we have that

$$\begin{aligned}\varphi(\mathfrak{a})\left(s^{-1}at^{-1}b - t^{-1}bs^{-1}a\right) &= \nu_{\{\mathfrak{a}\}}\left(\phi\left(s^{-1}at^{-1}b - t^{-1}bs^{-1}a\right)\right)\\ &\geq \gamma + \nu_{\{\mathfrak{a}\}}\left(\phi\left(s^{-1}at^{-1}b\right)\right)\\ &= \gamma + \varphi(\mathfrak{a})\left(s^{-1}at^{-1}b\right).\end{aligned}$$

Since $\{s^{-1}a : s \in \langle H_1 \rangle, a \in K[x,y]_q\}$ is dense in $\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_r)$ we can conclude that $\varphi(\mathfrak{a})$ is quasi abelian of level γ.
The estimate (4.5.1) implies that $\varphi(\mathfrak{a}) \in \operatorname{Spmqb}\left(\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_r)\right)$. □

Proposition 4.5.8. *Let $\mathcal{S} \subseteq \mathcal{T}_q$ be a finite subtree and let $r \in \mathbb{R}^2$ be such that $r_1 \geq r_2$ and that $\mathcal{S}_r \to \mathcal{S}$ is surjective.*
The map $K[x,y]_q \longrightarrow \mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_r)$ induces a map

$$\bar{\psi} : \operatorname{Spmq}\left(\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_r)\right) \longrightarrow \operatorname{Spmq}\left(K[x,y]_q\right)$$

which restricts to a map

$$\psi : \operatorname{Spmqb}\left(\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_r)\right) \longrightarrow U_r(\mathcal{S}).$$

ψ is the inverse of φ and both maps are level preserving bijections.

Proof. Let $\mathfrak{b} \in \operatorname{Spmqb}\left(\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_r)\right)$ and let $(a,s,r) \in A(\mathcal{S})$. Then there exists an $\mathfrak{a} \in \Lambda_{\mathcal{S}_{r_1,r_2}}$ with $\nu_{\mathcal{S}_r}\left(s^{-1}a\right) = \mathfrak{a}(a) - \mathfrak{a}(s)$. Since $\mathfrak{a} \in U_r(\mathcal{S})$ we can conclude that $\mathfrak{a}(a) - \mathfrak{a}(s) \geq -r$. Hence

$$\begin{aligned}\mathfrak{b}(a) - \mathfrak{b}(s) = \mathfrak{b}\left(s^{-1}a\right) &\geq \nu_{\mathcal{S}_r}\left(s^{-1}a\right)\\ &= \mathfrak{a}(a) - \mathfrak{a}(s) \geq -r\end{aligned}$$

and thus $\bar{\psi}(\mathfrak{b}) \in U(\mathcal{S})$. Since for $s \in H_1$ we have that

$$r_2 \leq \min\left\{\mathfrak{a}(s) : \mathfrak{a} \in \Lambda_{\mathcal{S}_{r_1,r_2}}\right\} = \nu_{\mathcal{S}_r}(s) \leq \mathfrak{b}(s)$$

and

$$-r_1 \leq \min\left\{-\mathfrak{a}(s) : \mathfrak{a} \in \Lambda_{\mathcal{S}_{r_1,r_2}}\right\} = \nu_{\mathcal{S}_r}\left(s^{-1}\right) \leq \mathfrak{b}\left(s^{-1}\right) = -\mathfrak{b}(s)$$

we can conclude that $\bar{\psi}(\mathfrak{b}) \in U_r(\mathcal{S})$. Hence we obtain a map

$$\psi : \operatorname{Spmqb}\left(\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_r)\right) \longrightarrow U_r(\mathcal{S}).$$

Using that $\{s^{-1}a : s \in \langle H_1 \rangle, a \in K[x,y]_q\}$ is dense in $\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_r)$ and that an element $\mathfrak{b} \in \mathrm{Spmqb}\left(\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_r)\right)$ therefore is uniquely defined by its restriction to $K[x,y]_q$ one can see that the maps φ and ψ are inverse to each other. $\square$

4.5.9. Let $\mathcal{S} \subseteq \mathcal{T}_q$ be a finite subtree and let $r \in \mathbb{R}^2$ be such that $r_1 \geq r_2$ and that $\mathcal{S}_r \to \mathcal{S}$ is surjective. We then have a canonical map

$$U_r(\mathcal{S}) \longrightarrow \mathcal{S}_r$$

by restricting an element $\mathfrak{a} \in U_r(\mathcal{S})$ to $K^2 \cong Kx \oplus Ky$. Combining this map with ψ we obtain a surjective map

$$r : \mathrm{Spmqb}\left(\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_r)\right) \longrightarrow \mathcal{S}_r$$

which is the analogue of the reduction map in the commutative case.
In the following we will have the same notation for an element $\mathfrak{a} \in U_r(\mathcal{S})$ and its image under φ.

Lemma 4.5.10. *Let $\mathcal{R} \subseteq \mathcal{S}$ be finite subtrees of $\mathcal{T}_q$ and let $r, t \in \mathbb{R}^2$ be such that $t_1 \geq r_1 \geq r_2 \geq t_2$ and that the canonical maps $\mathcal{S}_t \to \mathcal{S}$, $\mathcal{R}_r \to \mathcal{R}$ are surjective. Then there exists a canonical valuation increasing K-Banach algebra morphism*

$$\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_t) \longrightarrow \mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{R}_r).$$

Proof. This follows from the universal property of algebraic microlocalization. $\square$

Definition 4.5.11. Let $\mathcal{S} \subseteq \mathcal{T}_q$ be a subtree and let $(\mathcal{S}_n)_{n\in\mathbb{N}}$ be a sequence of finite subtrees of $\mathcal{T}_q$ such that $\mathcal{S}_n \subseteq \mathcal{S}_{n+1}$ and $\underset{n\in\mathbb{N}}{\cup} \mathcal{S}_n = \mathcal{S}$. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^2$ such that

1. $(\mathcal{S}_n)_{r_n} \to \mathcal{S}_n$ is surjective.
2. $r_{n+1,1} \geq r_{n,1} \geq r_{n,2} \geq r_{n+1,2}$.
3. $\underset{n\to\infty}{lim}(r_{n,1}) = \infty$ and $\underset{n\to\infty}{lim}(r_{n,2}) = -\infty$

Then we define the locally convex K-algebra

$$\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_e) := \varprojlim_n \mathcal{O}_{\mathcal{N}_{q,e}}((\mathcal{S}_n)_{r_n}).$$

Recall that $\mathcal{S}_e$ was defined to be the preimage of $\mathcal{S}$ under the canonical map $\mathcal{N}_e \to \mathcal{T}$.

Definition 4.5.12. Let $\mathcal{S} \subseteq \mathcal{T}_q$ be a subgraph with connected components $\mathcal{S}_i$, $i \in I \subseteq \mathbb{N}$. We then define

$$\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_e) := \prod_{i\in I} \mathcal{O}_{\mathcal{N}_{q,e}}\left((\mathcal{S}_i)_e\right).$$

For two subgraphs $\mathcal{R} \subset \mathcal{S}$ there is a canonical continuous algebra morphism

$$\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_e) \longrightarrow \mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{R}_e)$$

by Lemma 4.5.10.

4.5.13. Let $\mathfrak{S}$ be the category of all subgraphs of $\mathcal{T}_q$ with morphisms the inclusions maps. Let $\mathfrak{TA}$ be the category of locally convex K-algebras. Then

$$\mathcal{S} \mapsto \mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_e)$$

defines a contravariant functor $\mathfrak{S} \longrightarrow \mathfrak{TA}$ i.e. a presheaf.

4.6 Algebras attached to subtrees

Here we will describe a quantized analogue of the p-adic upper half plane.

Definition 4.6.1. Define the set

$$\mathcal{R} := \left\{ \begin{array}{c} s_1^{-1} g_1 \cdots s_n^{-1} g_n : n \in \mathbb{N}, \text{ and } g_i \in K[x,y]_q \text{ homogenous, } s_i \in H_1 \\ \text{and } \sum_i (\deg g_i - \deg s_i) = 0 \end{array} \right\}.$$

Lemma 4.6.2. *Let $\mathcal{S}$ be a finite subtree of $\mathcal{T}_q$ and let r_1, r_2 be such that $\mathcal{S}_{r_1,r_2} \to \mathcal{S}$ is surjective. Let $v := s_1^{-1} g_1 \cdots s_n^{-1} g_n \in \mathcal{R}$ and let $\sigma \in S_n$. Then there exist finitely many elements $u_i \in \mathcal{R}$ such that*

$$v = s_{\sigma(1)}^{-1} \cdots s_{\sigma(n)}^{-1} g_1 \cdots g_n + \sum_i u_i$$

and

$$\nu_{\mathcal{S}_{r_1,r_2}}(u_i) \geq \nu_{\mathcal{S}_{r_1,r_2}}\left(s_1^{-1} g_1 \cdots s_n^{-1} g_n\right) + l_{\Lambda_{\mathcal{S}_{r_1,r_2}}}.$$

Proof. We have that

$$x^{n_1} y^{m_1} x^{n_2} y^{m_2} = x^{n_2} y^{m_2} x^{n_1} y^{m_1} + \left(q^{-(n_2+m_1)} - q^{-(n_1+m_2)}\right) x^{n_1+n_2} y^{m_1+m_2}.$$

Thus for $f \in K[x,y]_q$ a homogeneous polynomial and $s \in H_1$ we can conclude

$$sf = fs + g$$

where g is either a homogeneous polynomial with $\deg(g) = \deg(f)+1$ or $g = 0$. Hence in $\mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_{r_1,r_2}))$ we have

$$fs^{-1} = s^{-1} f + s^{-1} g s^{-1}.$$

Similarly we for $s, t \in H_1$ there exists a homogeneous polynomial $g \in K[x,y]_q$ with $g = 0$ or $\deg(g) = 2$ such that $s^{-1}t^{-1} = t^{-1}s^{-1} + s^{-1}t^{-1}gt^{-1}s^{-1}$. Using the first consideration for $g_j s_{j+1}^{-1}$ we obtain

$$v = s_1^{-1} g_1 \cdots s_j^{-1} s_{j+1}^{-1} g_j g_{j+1} \cdots s_n^{-1} g_n + s_1^{-1} g_1 \cdots s_j^{-1} s_{j+1}^{-1} g s_{j+1}^{-1} g_{j+1} \cdots s_n^{-1} g_n$$

for some $g \in K[x,y]_q$ with $\deg(g) = \deg(f) + 1$ or $g = 0$. Hence

$$s_1^{-1} g_1 \cdots s_j^{-1} s_{j+1}^{-1} g s_{j+1}^{-1} g_{j+1} \cdots s_n^{-1} g_n \in \mathcal{R}.$$

By Proposition 4.5.4 we know that

$$\nu_{\mathcal{S}_{r_1,r_2}} \left(s_1^{-1} g_1 \cdots s_j^{-1} s_{j+1}^{-1} g s_{j+1}^{-1} g_{j+1} \cdots s_n^{-1} g_n \right) \geq \nu_{\mathcal{S}_{r_1,r_2}} (v) + l_{\Lambda_{\mathcal{S}_{r_1,r_2}}}$$

Repeating this process finitely many times we can conclude the claim. □

Lemma 4.6.3. *Let $\sum_i u_i \neq 0$ be a finite sum with $u_i \in \mathcal{R}$. Then there exists a finite sum $\sum_l v_l$ with $v_l \in \mathcal{R}$ and $\sum_i u_i = \sum_l v_l$ and*

$$\nu_{\mathcal{S}_{r_1,r_2}} \left(\sum_l v_l \right) = \min \left\{ \nu_{\mathcal{S}_{r_1,r_2}}(v_l) \right\}.$$

Proof. By Lemma 4.6.2 we know that there exist $s_1, ..., s_k \in H_1$, a homogeneous polynomial $f_i \in K[x,y]_q$ of degree k and $v_{i,j} \in \mathcal{R}$ such that

$$u_i = s_1^{-1} \cdots s_k^{-1} f_i + \sum_j v_{i,j}$$

with $\nu_{\mathcal{S}_{r_1,r_2}} \left(s_1^{-1} \cdots s_k^{-1} f_i \right) = \nu_{\mathcal{S}_{r_1,r_2}} (u_i)$ and

$$\nu_{\mathcal{S}_{r_1,r_2}} (v_{i,j}) \geq \nu_{\mathcal{S}_{r_1,r_2}} (u_i) + l_{\Lambda_{\mathcal{S}_{r_1,r_2}}}.$$

Note that we can use the same $s_1, ..., s_k \in H_1$ for every u_i by inserting elements of the form $s_i^{-1} s_i$ if necessary. Thus

$$\sum_i u_i = s_1^{-1} \cdots s_k^{-1} \sum_i f_i + \sum_i v_{i,j} \tag{4.6.1}$$

If

$$\nu_{\mathcal{S}_{r_1,r_2}} \left(s_1^{-1} \cdots s_k^{-1} \sum_i f_i \right) < \min \left\{ \nu_{\mathcal{S}_{r_1,r_2}}(u_i) \right\} + l_{\Lambda_{\mathcal{S}_{r_1,r_2}}} \leq \min \left\{ \nu_{\mathcal{S}_{r_1,r_2}}(v_{i,j}) \right\}$$

the right hand side of (4.6.1) is a desired.
If not we have found $w_{l,1} \in \mathcal{R}$ with $\sum_i u_i = \sum_l w_{l,1}$ and

$$\min \left\{ \nu_{\mathcal{S}_{r_1,r_2}}(w_{l,1}) \right\} \geq \min \left\{ \nu_{\mathcal{S}_{r_1,r_2}}(u_i) \right\} + l_{\Lambda_{\mathcal{S}_{r_1,r_2}}}.$$

Thus we can repeat the whole process with the $w_{l,1}$. Since $\sum_i u_i \neq 0$ after finitely many repetitions we obtain elements v_l as desired. $\square$

Lemma 4.6.4. *Let $\mathcal{S}$ be a finite subtree of $\mathcal{T}_q$. Let $g^{-1} \in G(q)$ and let $r \in \mathbb{R}^2$ be such that $\nu_{g,r} \in U(\mathcal{S})$. Then for every $\tilde{r} = (a, a) \in \mathbb{R}^2$ and every $f \in \mathcal{R}$ we have that*

$$\nu_{g,r}(f) = \nu_{g,r+\tilde{r}}(f).$$

Proof. Let $f = s_1^{-1} g_1 \cdots s_n^{-1} g_n$ and let $s := s_1 \cdots s_n$ and $e := g_1 \cdots g_n$. Then s and e are homogeneous polynomials in $K[x, y]_q$ of the same degree $k \in \mathbb{N}_0$. Hence $\nu_{g,r}(s) = \nu_{g,r+\tilde{r}}(s) - ak$ and $\nu_{g,r}(e) = \nu_{g,r+\tilde{r}}(e) - ak$. Hence we obtain

$$\begin{aligned}
\nu_{g,r}(f) &= \nu_{\{\nu_{g,r}\}}\left(s_1^{-1} g_1 \cdots s_n^{-1} g_n\right) \\
&= \nu_{\{\nu_{g,r}\}}\left(s^{-1} e\right) \\
&= \nu_{g,r}(e) - \nu_{g,r}(s) \\
&= \nu_{g,r+\tilde{r}}(e) - ak - \nu_{g,r+\tilde{r}}(s) + ak \\
&= \nu_{g,r+\tilde{r}}(e) - \nu_{g,r+\tilde{r}}(s) \\
&= \nu_{\{\nu_{g,r+\tilde{r}}\}}\left(s^{-1} e\right) \\
&= \nu_{\{\nu_{g,r+\tilde{r}}\}}\left(s_1^{-1} g_1 \cdots s_n^{-1} g_n\right) \\
&= \nu_{g,r+\tilde{r}}(f).
\end{aligned}$$

$\square$

Corollary 4.6.5. *Let $\mathcal{S}$ be a finite subtree of $\mathcal{T}_q$ and let $f \in \mathcal{R}$. Let $r, \tilde{r} \in \mathbb{R}^2$ be such that $\mathcal{S}_r \to \mathcal{S}$ and $\mathcal{S}_{\tilde{r}} \to \mathcal{S}$ are surjective. Then we have $\nu_{\mathcal{S}_r}(f) = \nu_{\mathcal{S}_{\tilde{r}}}(f)$.*

Definition 4.6.6. Let $\mathcal{S}$ be a finite subtree of $\mathcal{T}_q$ and let $r \in \mathbb{R}^2$ be such that $\mathcal{S}_r \to \mathcal{S}$ is surjective. We define

$$\mathcal{O}_{\mathcal{T}_q}(\mathcal{S}) := \left\{ \sum_{i \in \mathbb{N}} g_i \in \mathcal{O}_{\mathcal{N}_{q,e}}(\mathcal{S}_e) : g_i \in \mathcal{R} \text{ and } \lim_{i \to \infty} \nu_{\mathcal{S}_r}(g_i) = \infty \right\}.$$

By Corollary 4.6.5 this definition does not depend on r.

Lemma 4.6.7. *Let $\mathcal{S} \subseteq \mathcal{T}_q$ be a finite subtree and let $r \in \mathbb{R}^2$ be such that $r_1 \geq r_2$ and $\mathcal{S}_r \to \mathcal{S}$ is surjective. Then for every $f \in \mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$ there exist $g_i \in \mathcal{R}$ with $f = \sum_i g_i$, $\lim_{i \to \infty} g_i = 0$ with respect to $\nu_{\mathcal{S}_r}$, and*

$$\nu_{\mathcal{S}_r}(f) = \min \left\{ \nu_{\mathcal{S}_r}(g_i) \right\}.$$

Moreover $\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$ together with $\nu_{\mathcal{S}_r}$ is a K-Banach algebra.

Proof. The first claim follows from Lemma 4.6.3. To show the second let $a_i = \sum_j g_{i,j}$ be a sequence with $\lim_{i \to \infty} a_i = 0$. By the first statement of the Lemma we can assume that $\nu_{\mathcal{S}_r}(a_i) = \min \{\nu_{\mathcal{S}_r}(g_{i,j})\}$. But that implies that

$$\lim_{i+j \to \infty} g_{i,j} = 0$$

and hence $\sum_{i,j} g_{i,j} \in \mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$. Thus $\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$ is complete with respect to $\nu_{\mathcal{S}_r}$. □

Corollary 4.6.8. *For $r, t \in \mathbb{R}^2$ such that $\mathcal{S}_r \to \mathcal{S}$ and $\mathcal{S}_t \to \mathcal{S}$ are surjective, the valuations $\nu_{\mathcal{S}_r}$ and $\nu_{\mathcal{S}_t}$ coincide on $\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$.*

Proof. By the submultiplicativity of $\nu_{\mathcal{S}_r}$, the space $\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$ is a K-Banach algebra with valuation $\nu_{\mathcal{S}_r}$. Choose $z \in \mathbb{R}^2$ such that $\mathcal{S}_z \subseteq \mathcal{S}_r \cap \mathcal{S}_t$ and $\mathcal{S}_z \to \mathcal{S}$ is surjective. Let $a \in \mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$. By Lemma 4.6.7 there exist $g_i \in \mathcal{R}$ with $a = \sum_i g_i$ and

$$\nu_{\mathcal{S}_z}(a) = \min\{\nu_{\mathcal{S}_z}(g_i)\}$$

Hence by Corollary 4.6.5

$$\nu_{\mathcal{S}_z}(a) \geq \nu_{\mathcal{S}_r}(a) \geq \min\{\nu_{\mathcal{S}_r}(g_i)\} = \min\{\nu_{\mathcal{S}_z}(g_i)\} = \nu_{\mathcal{S}_z}(a).$$

Thus we can conclude that $\nu_{\mathcal{S}_z} = \nu_{\mathcal{S}_r}$ and analogously $\nu_{\mathcal{S}_z} = \nu_{\mathcal{S}_t}$. □

4.6.9. For $\mathfrak{a} \in \operatorname{Spmq}\left(\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})\right)$ we can define a semivaluation on K^2 by

$$\alpha_{\mathfrak{a}}(ae_1 + be_2) := \mathfrak{a}\left(x^{-1}(ax + by)\right)$$

where e_1, e_2 is the standard basis of K^2. Thus we obtain a map

$$\operatorname{Spmq}\left(\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})\right) \longrightarrow \mathcal{N}_e$$

by sending $\mathfrak{a}$ to $[\alpha_{\mathfrak{a}}]$. Since $s \in H_1$ is invertible in $\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$ we can conclude that $\alpha_{\mathfrak{a}}$ is a valuation. Let $\mathfrak{a} \in \operatorname{Spmqb}\left(\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})\right)$ and let $(a, s, t) \in A(\mathcal{S})$ and let $r \in \mathbb{R}^2$ be such that $r_1 \geq r_2$ and $\mathcal{S}_r \to \mathcal{S}$ is surjective. Then there exists $\lambda \in U_r(\mathcal{S})$ such that $\nu_{\mathcal{S}_r}(s^{-1}a) = \lambda(a) - \lambda(s)$. Hence

$$\begin{aligned}\alpha_{\mathfrak{a}}(a) - \alpha_{\mathfrak{a}}(s) &= \mathfrak{a}\left(s^{-1}a\right) \geq \nu_{\mathcal{S}_r}\left(s^{-1}a\right) \\ &= \lambda(a) - \lambda(s) \geq -t\end{aligned}$$

since $\lambda \in U_r(\mathcal{S})$. This implies $\alpha_{\mathfrak{a}} \in \mathcal{S}$.

4.6.10. The considerations in **4.6.9** imply that we have a map

$$r : \operatorname{Spmqb}\left(\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})\right) \longrightarrow \mathcal{S},$$

which is the analogue of the reduction map of $\mathcal{H}$. By **4.4.2**, Proposition 4.5.7 and Lemma 4.6.4 this map has a section

$$\mathcal{S} \longrightarrow \operatorname{Spmqb}\left(\mathcal{O}_{\mathcal{T}_q}(\mathcal{S})\right).$$

4.6.11. For two finite subtrees $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{T}_q$. Then by Lemma 4.5.10 we have a canonical valuation increasing K-Banach algebra morphism

$$\mathcal{O}_{\mathcal{T}_q}(\mathcal{S}_2) \longrightarrow \mathcal{O}_{\mathcal{T}_q}(\mathcal{S}_1).$$

Definition 4.6.12. Let $\mathcal{S} \subseteq \mathcal{T}_q$ be a subtree and let $\mathcal{S}_i$ be an increasing sequence of finite trees such that $\cup_i \mathcal{S}_i = \mathcal{S}$. Then we define

$$\mathcal{O}_{\mathcal{T}_q}(\mathcal{S}) := \varprojlim_i \mathcal{O}_{\mathcal{T}_q}(\mathcal{S}_i).$$

Let $\mathcal{S} \subseteq \mathcal{T}_q$ be a subgraph and let $\{\mathcal{S}_k\}_{k \in I}$ be the set of its connected components. We then define

$$\mathcal{O}_{\mathcal{T}_q}(\mathcal{S}) := \prod_k \mathcal{O}_{\mathcal{T}_q}(\mathcal{S}_k).$$

4.6.13. For two subgraphs $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{T}_q$ we have a canonical continuous morphism of locally convex K-algebras

$$\mathcal{O}_{\mathcal{T}_q}(\mathcal{S}_2) \longrightarrow \mathcal{O}_{\mathcal{T}_q}(\mathcal{S}_1).$$

The assignment

$$\mathcal{S} \mapsto \mathcal{O}_{\mathcal{T}_q}(\mathcal{S})$$

defines a contravariant functor $\mathfrak{S} \longrightarrow \mathfrak{TA}$. This functor is our quantized p-adic upper half plane.

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Selbständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß §7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 angegebenen Hilfsmittel angefertigt habe.

Berlin, den 6. Dezember 2016 Christian Wald